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Constructing Tests of Mathematical Concepts for Young Children

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About ten years ago, the Mathematics Program in the elementary schools of New York City was modified so as to place an increasing emphasis on the ability of young children to do mathematical thinking. It was assumed that children can learn to think in the abstract terms of mathematics only if they learn meaningful concepts of arithmetic, or mathematics. The mathematics to be learned must not be too difficult for pupil understanding and must not be too easy to preclude a challenge to thinking.

Developmental Mathematics, or Arithmetic

The program in the elementary schools is developmental. The concepts the child acquires are developmental. They range from vague, approximate, and descriptive levels—for example, concepts of far, near, and many to the more specific and exact concepts of 23 feet, $19\frac{1}{2}$ pounds, or two 2's are 4, whether added or multiplied. The mathematics is derived from the children's experiences. The mathematical materials are used so that mathematics in the child's experiences can be isolated and learned in mathematical ways. The initial understandings of mathematical concepts grow out of such experiences as block building, painting, planting bulbs, using puzzles, planning parties, or preparing food. In such experiences, the children are taught to observe the comparative amounts of water and paints in jars, the shapes of puzzle pieces, the heights of children, the space needed for tables and chairs, the weights of boxes. Terms used to describe such comparisons include: higher, heavier, closer, longer, warmer, etc.

Sequential Development of Arithmetic Concepts

Each mathematics topic or concept is built through sequential developmental levels. First, the children engage in experiences where the number content is both directly or indirectly learned under the direction of the teacher. Second, they use representative materials—such as beads, discs, dimes and pennies, and tens frames—to establish relationships among numbers and arithmetical concepts. Third, under the direction of the teacher, they are stimulated to think through mathematical relationships and find different ways of solving the same problem. The fourth step is the written computation in which the children deal with abstract number relationships.

Specifications for a Test of Arithmetical Meanings

After the arithmetic curriculum for young children was determined, the need was apparent to build a test that would help teachers to estimate how adequately the pupils were gaining the concepts or meanings. A committee of teachers of Developmental Mathematics cooperated with the test technicians from the Bureau of Educational Research in order to construct such tests. A first step in this process was to set up the specifications for the tests. These specifications defined the scope of each test and nature of the test item content.

Grade 1 Test

At Grade 1 level, the test was divided into two major parts. Part 1 dealt with pre-measurement concepts and Part 2 with numerical concepts. A brief outline of the specifications follows:

Part 1 includes questions measuring the child's knowledge of pre-measurement concepts in the following areas:

1. Size—largest, longest, etc.
2. Shape—circle, square, etc.
3. Weight—heaviest, lightest, etc.
4. Time—longest, "right away," etc.
5. Indefinite Quantity—most, few, etc.
6. Place and Distance—above, outside, etc.

Part 2 includes questions measuring the child's mastery of mathematical concepts in the following areas:

1. Use of Symbols to Express One-to-One Correspondence—3 for *three*, 6 for *six*, etc.
2. Cardinal Numbers as Expression of Total Groupings—wheels of an automobile, pennies in a nickel, etc.
3. Use of Quantitative Terms—double, pair, etc.
4. Ordinal Numbers as Expression of Place in a Series—second, fourth, etc.
5. Addition—components of 3 (1 and 2); adding 2, with 4 as point of reference, etc.
6. Subtraction—remainder (6-1), comparison—difference (10-5), etc.
7. Multiplication—Two 2's, four 2's, etc.
8. Division—twos in eight, fives in ten, etc.
9. Fractional Parts—half of single object, half of group of six, etc.

Grade 2 Test

The test for Grade 2 was also divided into two parts. Part 1 was designed to measure pre-measurement concepts and standard measures; Part 2 to measure numbers and processes. An outline of the scope and nature of the content of the Grade 2 test follows:

Part 1 includes questions measuring the child's knowledge of pre-measurement concepts and standard measures in the following areas:

1. Size—largest, etc.
2. Shape—rectangle, etc.
3. Weight—lightest, etc.
4. Indefinite Quantity—empty, etc.
5. Place and Distance—center, right, etc.
6. Measures—clock, calendar, etc.

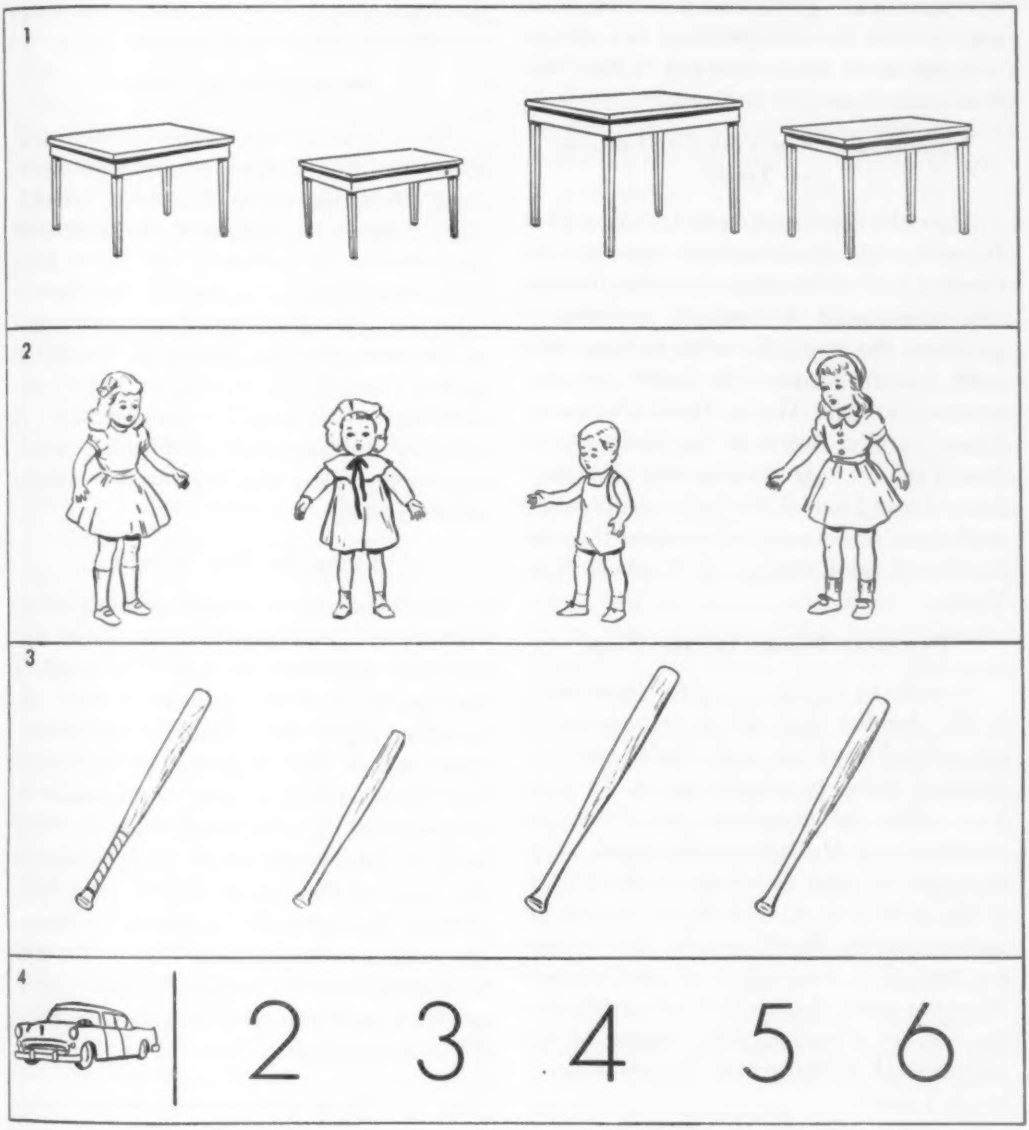
Part 2 includes questions measuring the child's mastery of numbers and processes in the following areas:

1. Use of Ordinal Numbers as an Expression of Place in a Series—fourth, eighth, etc.
2. Fractional Parts—one third of single object, half of group of 10, etc.
3. Addition—components of 7 (3 and 4); three addends, 5, 4, 3, etc.
4. Subtraction—remainder (8-5), remainder (12-6), etc.
5. Multiplication—three 3's, four 2's, etc.
6. Division—twos in eight, twos in twelve, etc.

Preparing Test Items

The Developmental Mathematics teachers cooperated with the Bureau of Educational Research in preparing test items, or exercises. Approximately two times as many items were prepared as were used in the final edition of the test. The premeasurements concepts are measured by drawings of such objects as a series of four tables, four dolls, or four baseball bats in a panel or row. The child is directed to mark a cross on (1) the *smallest* table in the row, (2) the *largest* doll in the row, or (3) the *longest* bat of four in a row. Numerical concepts are measured by a drawing of an object followed by a row of five numbers. For example, a drawing of an automobile is followed by the numbers: 2 3 4 5 6.

The child is directed to look at the numbers beside the drawing of the automobile. He is asked, "How many wheels does an automobile have? Make a cross on that number." It will be observed that the presentation of test items is largely of a pictorial nature for measuring the number concepts of young children.



Tryout of Test Items

The next step in the construction of the tests was a try-out of the items using a group of 200 children. The results of this try-out permitted an analysis of the items and indicated which items were appropriate without change, which ones had to be revised and which items were not suited to the test situation. After the preliminary try-out and revision of the tests, a second try-out with a group of about 300 children was arranged. For each item an index of difficulty and discrimination, using the usual biserial coefficient, was obtained. A selection of the test items was then made and the test was administered to a sample population of approximately 1,000 children both in grade 1 and grade 2.

National and New York City Editions of Tests

After the construction of the *New York Inventory of Mathematical Concepts* for Grade 1 and for Grade 2, increasing interest was manifested by school systems to purchase the tests. In order to meet this need, parallel forms of the tests were constructed for the World Book Company. These parallel forms of the test entitled *New York Test of Arithmetical Meaning*, Level 1 and Level 2, for the corresponding grade levels may now be obtained through the World Book Company, Yonkers, New York.

Percentile Norms for the Tests

In order to obtain norms for these tests, it was decided that the most meaningful interpretation of the test results could be obtained using percentile scores. In New York City the percentile scores for the *Inventories of Mathematical Concepts* were obtained by administering the final form of the test to a representative sample of approximately 20,000 pupils. The norms for the *New York Test of Arithmetical Meaning* were obtained by administering the test to a nation-wide sample of approximately 20,000 pupils at each of grade levels 1 and 2.

Reliability of the Tests

The accuracy of each test as an instrument for measuring arithmetical meanings and concepts was determined by means of the Kuder-Richardson and the Spearman-Brown formulas. In addition, the probable error of measurement of an obtained raw score was determined. The median *coefficient of reliability* for Grade 1 total score is .88. The *probable error* of measurement is approximately 2 raw score points. On the test for Grade 2 similar studies of reliability provide a median *coefficient* of about .88 for the total test. The *probable error* of measurement is approximately 2 raw score points.

Validity of the Tests

The validity of the tests is based mainly on a content analysis of the courses of study of mathematics in grades 1 and 2. They represent a sample of the content of the courses. In addition, the items have been examined by a jury of teachers in order to determine their representiveness of the concepts and meanings taught in grades 1 and 2. The tests show a high correlation with pupils' performance in learning mathematics in primary grade classrooms. The tests meet these criteria satisfactorily.

Using the Test Results

The tests may be used to provide an inventory of the arithmetical meanings and concepts acquired by pupils in grade 1 and grade 2. They provide a basis for grouping pupils for arithmetic instruction. They aid in the diagnosis of individual and class strengths and weaknesses in arithmetical meanings and concepts. They help to adapt arithmetic instruction to the level of the group and to each individual. They provide one basis for assessing progress in arithmetic instruction and for interpreting the arithmetic program in grades 1 and 2 to parents and patrons of the school system.

(*Editor's Note on page 108*)

New Developments in Arithmetic Teaching in Britain

Introducing the Concept of "Set"

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1. Introduction

If one is prepared to study the activity of the mathematician, not in its final form, but as it actually takes place, there are many new discoveries to be made. By the adoption of this view point, the problems met in the teaching of mathematics at all levels are seen in a completely new light. In this short paper we shall first consider the place of arithmetic within the general mathematical activity, and then summarise a method developed by the author with material due to the genius of a Belgian schoolmaster, Georges Cuisenaire.

2. Characterisation of the Mathematical Activity

While it is true that every mathematician can recognise what is mathematical in the treatment of any subject matter, no definition of mathematics which would be acceptable to all mathematicians can be proposed. Our attempt to clarify this somewhat baffling position has led us to investigate what it is that characterises the activity of the mathematician, and we have found it to be an awareness of a particular type. We no longer need to contrast mathematical activity with other kinds of activity; we only need to state what it is and every mathematician will recognise that this is in fact what he experiences when he is at work.

The first important point is to recognise that there are what we shall call *virtual actions*. Every human being is involved in a multitude of actions. When he stops to consider a position inwardly, for the purpose of selecting some aspect of an action as being preferable to others and deciding to embark upon it, he is substituting virtual actions for real ones. Moreover virtual actions may lead to the extension of actual actions when these are replaced by internal processes. For instance, stringing beads is an action, and to imagine oneself doing so is, at first, to evoke the gesture without actually performing it, but to become aware of it as a possible action that can recur indefinitely is the virtual action which will serve as basis for the indefinite extension of addition of units.

Virtual actions participate of actions and therefore of all biological and mental life. In them, perception is implicit and with it the dynamics of the mind. By becoming virtual, action extends its range, but the extension is not arbitrary. There are certain constraints inherent in the original action, and these now become potential. Thus, if for the actual staircase up which we go we substitute the set of steps, the virtual actions we obtain will be arithmetical progressions, and nothing more. In fact, the structures contained in the virtual action limit their extension, and it is awareness of these structures that makes the mathematician.

All those, then, who are capable of replacing actual actions by ones that are virtual and contemplating the structures therein contained, act, when they do so, as mathematicians. All of us are of course so capable, but not everyone delights in that kind of exercise, and so all men, although potential mathematicians, do not become professionals or even amateurs in this field. Life offers many other attractions, both through actual and virtual actions, which involve awareness of other things than structures.

3. Structures and Relationships

In the last fifty or so years, mathematicians have come to see that they are essentially concerned with sets of elements upon which they *induce* structures by means of relationships. What matters to them is to know the maximum field of validity of a statement and to discover the proofs which most adequately correspond to the propositions involved. They state, in fact, that the concept of "mathematical being" can be no more than a mental construct indicating a momentary halt in the process of discovering what lies behind a complex structure.

But what mathematicians contemplate are *mathematical situations* containing a number of structures and relationships, but themselves defined by certain relationships. There is an indefinite number of such situations. As an example, we can take the "geo-boards" which we have introduced as a simple but pregnant means to understanding of what constitutes a geometrical situation. A board on which a lattice is marked can provide, when points of the lattice are joined, a number of figures (polygons). These can be described and their number found. By concentrating on particular aspects of the figures, we can select those which are congruent or similar, those which are symmetrical or regular. We can calculate areas or look for concurrencies, etc. The situation is mathematical by the fact that we are concerned with certain of the existing relationships, and not with the wood, the nails and the elastic bands which are the essential elements of the actual board. By our actions we create relationships in the situation, and we are doing mathematics since we are abstracting one or more relationships from the background of all the possible relationships it contains.

To differentiate between structures and relationships we could say that we are more easily aware of the latter and that when relationships become fixed in our minds we call them structures. In relationships we are aware of the dynamic element; when it is overshadowed we see them as structures.

Now circles are relationships, and we can conceive of relationships between circles, e.g. overlapping, cutting, concentricity, and from these we can extract new relationships, such as, for example, that the common chord of cutting circles is perpendicular to the line of the centres.

The vision of a mathematical eye is of these things, and it is a vision that can become available to others if they are made aware that they are aware of relationships. To our mind, it is precisely this that is the function of mathematics teaching.

4. Arithmetic

What awareness is peculiar to arithmetic? How are arithmetical situations created?

It is obvious that in arithmetic we are always concerned with numbers. The set on which we operate is a set of numbers, and the relationships that underlie the set are those which constitute what we can call "qualitative arithmetic." First we have the awareness of sets and sub-sets, these being formed of the elements satisfying a certain relationship. Thus "equivalent relations" (binary relations satisfying the conditions of being reflexive, symmetrical and transitive), sub-divide a set into sub-sets or classes of equivalence such that each element belongs to one class and one only. By selecting one element from each class we obtain what is called the "quotient-set." "Order relations" (binary relations which are transitive) structure some sets, and are more primitive than numbers. But the dynamic cause underlying arithmetic is what is called the "algebraic structures." The set is such that "operations" exist by which there corresponds to each pair of its elements another of its elements. The way in which the correspondence is made forms a set of relationships fixing the algebraic structures. A very important such

structure is the "group," which is recognised very early in childhood, and to which we shall return in a moment.

The Cuisenaire material consists of coloured rods 1 sq. cm. in section, varying in length from 1 to 10 centimetres, those of the same length being of the same colour and related lengths being of related colours (reds 2, 4, 8 cms; blues 3, 6, 9 cms; yellows 5, 10 cms.; white 1 cm.; black 7 cms.). A set of these rods is at once seen to be sub-divisible into classes of equivalence (the same colour or the same length), its quotient-set being formed of ten rods, one of each colour. The whole set is ordered (the order relation being "greater than or equal to" or "smaller than or equal to") as is also the quotient-set, but the latter is "strictly" ordered ("greater or smaller than"). The way in which the rods are cut allows of the introduction of an algebraic operation corresponding to the action of placing rods end to end and thus obtaining new lengths; this can be noted as addition. In this way we obtain a set which, if we suppose the existence of an indefinite number of rods, is "isomorphic" to that of the positive rationals, i.e. in a one-to-one correspondence with it, and such that the operations performed on a pair of corresponding elements in the rods and the rationals give results which also correspond. This last statement requires proof.

Since the length of any rod or of any number of rods end to end is a multiple of the white one, we obtain the sub-set of rationals constituted by the integers. But if we choose to measure any length using as measure any rod or length, we generate pairs of integers which are for us true fractions. The operation of addition of the integers satisfies the same conditions as the operation of putting rods end to end and finding the total length. In the case of fractions this will become clear when we have taken one further step in the direction of arithmetical situations.

With the Cuisenaire material, any given length can be constructed by a number of groupings of rods of other lengths. These groupings we call *decompositions* of that length, and it can be seen that the length is not affected by any permutation of the rods in a decomposition (we call this the commutative property of addition). By removing first one rod and then another, and discovering, through knowledge of what remains and of the original length, which rods were removed, we see that each addition corresponds to several subtractions. If the rods in a decomposition are of one colour only, we obtain at the same time two factors, which are sometimes equal, of the number represented by the length, and hence several fractions of it and their value. If the length cannot be completed by repetition of one rod or group of rods, we obtain division sums. When the set of decompositions contains no row of one colour (or a row found by the repetition of the same group), the number is prime.

A set of decompositions is therefore a fruitful arithmetical situation. The revolutionary character of the influence of the Cuisenaire material on arithmetic teaching is clear to see. In the first place it brings modern mathematics into the primary stages of schooling, in particular the important recognition that fractions are ordered pairs of integers. Secondly, it substitutes for the study of numbers the study of the sets of their decompositions, making apparent the dynamic operations that structure these sets. As a result, addition, subtraction, multiplication, factors, fractions and division are all seen at the same time, and as being generated by the virtual actions of the mind on the situations. Thirdly, through the presentation of isomorphic systems, results which are seen to be obvious in one of them appear as true also for the other. Colour and length offer information that is perceivable and its translation into numerical language gives the latter the dynamics which is far less apparent when notation only is used. Fourthly, since the rods are not sub-divided, they can represent a

different value each time they are used as measure for comparison with other rods. It is this fact that gives the material its unique property of introducing at the same time whole numbers and fractions, integers being seen as fractions whose unit of measurement is omitted and ignored.

All this is, of course, of tremendous significance in the learning and teaching of arithmetic. The several thousand classes in which the Cuisenaire material has been in use in the last few years testify to the fact that what earlier seemed impossible can now be done with ease. Children of 6 or 7 are thoroughly familiar with their tables, children of 5 conceive and compare fractions easily and accurately, children of 8 solve simultaneous equations and at 10 they understand permutations and combinations which they themselves form and analyse.

A detailed account of our work with this material in several countries has been given in other publications and the reader who is interested can refer to the bibliography that follows. The research stimulated by the introduction into schools of Cuisenaire's numbers in colour can be measured by the necessity felt for the creation of new international journals for the publication of material resulting from their use.

5. Conclusion

It has not been possible in this short note to elaborate any of the points we have made. This could indeed be done only in the form of a book. It may be felt that what is ultimately of primary importance is the raising of the level of teaching in schools, and that modern mathematics cannot be introduced into the syllabus of the early years while teachers of the first grades are insecure in their own knowledge of what they have to teach. This situation is universal and we have found that we can give to teachers, in intensive refresher courses of one to two weeks during which we use with them the material they will use on their return to school, the training that will enable them to recast the content of their syllabus and their methods of teaching. The ready response we have had from teachers in primary schools in several countries gives grounds for optimism and supports our hope that as a result of our sustained effort we shall in a few years see arithmetic enjoying a popularity equal to that it, at present, has with number fans.

We now know where the trouble lies and where the remedy is to be found. Pupils of all ages and ranges of ability have responded to our teaching, which takes them to a much higher level and is much more abstract than is usually the case, with joy and real understanding of the issues. Other teachers who, like us, have had faith in their pupils and dared to stretch their minds, have had a similar experience. We are no longer at the trial stage and can look to the future with confidence.

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N.B.—Cuisenaire's numbers in colour are world copyright and cannot be produced in the U.S.A. unless agreement is reached with the author of this article on behalf of Cuisenaire.

EDITOR'S NOTE. When Professor Fehr of Teachers College, Columbia University was in Europe last year he learned about the work of Dr. Gattegno both in England and on the continent. He also saw the colored rods which were developed by Georges Cuisenaire and which are described by Dr. Gattegno in this article. These are a different type of objective material than we commonly use in this country and in Europe Mr. Cuisenaire's methods seem more mathematical and more rapid than we find in use with our pupils in the early elementary school grades. Perhaps we do not exact as much thinking as we could and should from our pupils. With many kinds lf materials mathematical relationships can be discovered but it is the teacher who must provide the atmosphere favorable to thinking and discovery.

Dr. Gattegno devoted his early study to research questions in pure mathematics but soon turned his interest to the problems of teaching mathematics. In 1950, he organized international seminars in mathematics teaching and has demonstrated teaching in the elementary and secondary schools of the principal countries of Europe. He was a member of the British Association Committee on the Teaching of Mathematics in the Primary School. The bibliography will enable readers to become better acquainted with his work.

BOOK REVIEW

Math. Can be Fun. Teacher Edition, $2.50 and Student Edition, $2.00 by Louis Grant Brandes. J. Weston Walch, Publisher, Box 1075, Portland, Maine, 1956.

Here is an excellent collection of interesting mathematics which every teacher should have available. The author directs his materials primarily to students in grades 7, 8, and 9 but many exercises can be used as low as grade four. The teacher Edition contains the answers which have been omitted from the Student Edition. The material is presented in 200 pages of size 8 1/2 by 11 inches in offset print. There are many good and some amusing illustrations.

Mr. Brandes has dressed many of the old standard puzzles in new garb which is more appealing to youngsters. The content is organized under the following headings: (1) Number Oddities, (2) Puzzles, (3) Tricks and Games, (4) Facts and Stories, (5) Test Yourself, (6) Optical illusions, (7) Some Tough Nuts to Crack, and (8) Linkages. An appendix contains a bibliography and suggestions for teachers. The index not only lists the many topics but also classifies them as puzzles, stories, etc.

This is not intended to be a textbook. Rather, it should serve as interesting supplemental mathematics. Many people, both young and old, are intrigued by puzzles, games, tricks, oddities, and illusions and such devises often have a value far beyond the puzzle element. Many teachers have found that pupils gain an interest in mathematics through such devices. Here in one book which is addressed to pupils is a goldmine for the busy teacher.

BEN A. SUELTZ

Classroom Experiences with Recreational Arithmetic

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MUCH OF THE SO-CALLED DISLIKE for arithmetic in the intermediate grades is only pretense. Attitudes of apathy and hatred can be changed to contagious enthusiasm when methods of recreational arithmetic are employed. When pupils participate in the healthy enjoyment of number games, they are not ashamed to admit their enjoyment to the world in general. Original puzzles, tricks, and recreational units devised by pupils prove that working with numbers is not really distasteful.

Recreational mathematics can serve as a "pepper-upper" to start a school day or to begin an arithmetic period. A short impromptu game will sharpen wits on a dull day and will relieve tensions and boredom. Sometimes between periods, just before lunch, or before dismissal time there are five or ten minutes which seem to "dangle" and which are opportune for a game which can build a worthwhile interest in working with numbers. An occasional interruption of the usual routine with a puzzle, joke, story, riddle, or game not only provides a pleasurable break but may also be stimulating for interest in learning. There are many stories, games, and puzzles with a mathematical content.

In our area some children must come to school early and have free time before school and must stay indoors when weather does not permit outdoor play. A "number oddity" or a number progression or some pattern of numbers written on the chalk board will fill the pupil's time pleasantly and to good advantage.

HERE ARE A FEW EXAMPLES:

1. Select a number less than 10
   Multiply it by 9
   Use the result as a multiplier of 123456789
   (The final product will be a succession of the digit originally chosen)
   (Expect for a zero in the 10's place.)

$$\begin{array}{r} 7 \\ \times 9 \\ \hline 63 \end{array} \qquad \begin{array}{r} 123456789 \\ \times 63 \\ \hline 370370367 \\ 740740734 \\ \hline 7777777707 \end{array}$$

2. Select a three-digit number
   reverse the order of digits
   subtract the smaller number from the larger
   divide the result by
   (The answer can be read backward or forward)

$$\begin{array}{r} 453 \\ -354 \\ \hline 99 \end{array} \div 9 = 11$$

3. Add:

$$\begin{array}{r} 123456789 \\ 987654321 \\ 123456789 \\ 987654321 \\ 2 \\ \hline 2222222222 \end{array}$$

   (Don't show the answer, let pupils find it.)

Boys and girls like to see for themselves how these exercises "work out." They will test one repeatedly, try it with other

* Mrs. Nies developed an interest in and a basis for recreational arithmetic in a course with Dr. Margaret F. Willerding at Harris Teachers College, St. Louis, Mo.

numbers, and in the meantime they are developing speed and accuracy without sensing any drill. They think it is "kinda fun." Some days I will put a number oddity or riddle on the chalk board or bulletin board and make no comment about it. Many children will "surprise" me by solving the oddity in their spare time. Like many teachers, I have a stock of seasonal items. Occasionally codes are used and both the boys and girls have a great fondness for them.

1. CODE PROBLEM:

Add: In the sum each letter represents a distinct digit. Find the numbers.

	S	E	N	D
	M	O	R	E
M	O	N	E	Y

2. HOCUS POCUS HALLOWEEN:

Add. In the sum each letter represents a distinct digit. Find the numbers.

	H	O	C	U	S
	P	O	C	U	S
P	R	E	S	T	O

3. APRIL FOOL'S PUZZLE:

Write a number of three digits on your paper. Be sure that the difference between the first and last digit exceeds one. Reverse the digits. Find the difference between the two numbers. Reverse the digits of this difference. Add these two numbers. Multiply by a million. Subtract 966,685,433. Substitute these letters for figures: under every figure 1, write the letter L, under every figure 2, write the letter O, under every figure 3, write the letter F, under every figure 4, write the letter I, under every figure 5, write the letter R, under every figure 6, write the letter P, under every figure 7, write the letter A. Read the result backwards.

I have observed that frequently pupils will take a greater interest in numbers and measures if they know something about how these originated. The history of mathematics in a simplified version fascinates the elementary school child. My pupils particularly like such things as the "stick method" of counting, ancient ways of telling time, early calendars, etc. These topics also help them to appreciate modern arithmetic. The formula for finding area seems more useful to a child when he sees how troublesome area problems were to the Egyptian rope-stretchers. The story of our modern standardized measures and how these were developed can be both dramatic and worthwhile. The humor of impractical methods of measuring yards and feet in the days of King Henry II helps emphasize the importance of accuracy.

The interest of almost all children can be captured by stories, but some will prefer action. When pupils experiment with an abacus, geometric paper folding, or a set of Napier's "bones" which they themselves construct, I find their concepts of numbers and processes strengthened. Certain games and puzzles afford action, but not every child will enjoy arithmetic games immediately. He may not "catch on" to the tricks and short computations or he may not "see" a pattern or relationship in number progressions, or he may not grasp the essential point of a riddle. On the other hand, there are times when recreational materials will help a child to sense number relationships which he cannot understand through basic techniques. In certain remedial cases, when all other methods have failed, "light has dawned" during use of recreational devices. But it would be a mistake to force a child to participate in number games. Let him observe passively, he may soon take an active interest along with his classmates.

Likewise, it is a mistake to "explain" some short computations which may only confuse some of the children. A trick or a puzzle which calls for processes beyond the current learning level of the class is not useful. Conversely, one which

makes application of a child's current stock of number facts provides him with a thrill when he sees it work. The fourth grader who sees for the first time the interesting patterns made by multiples of 9 is actually gleeful when he reaches grades five and six and more advanced patterns he can work out with "Magic 9," "Tricky 3," and "Lucky 7."

TRICKY 3

37 × 3 = 111
37 × 6 = 222
37 × 9 = 333
37 × 12 = 444
37 × 15 = 555
37 × 18 = 666
37 × 21 = 777
37 × 24 = 888
37 × 27 = 999

LUCKY 7

15,873 × 7 = 111,111
15,873 × 14 = 222,222
15,873 × 21 = 333,333
15,873 × 28 = 444,444
15,873 × 35 = 555,555
15,873 × 42 = 666,666
15,873 × 48 = 777,777
15,873 × 56 = 888,888
15,873 × 63 = 999,999

MAGIC 9

123456789 × 9 = 111111111
123456789 × 18 = 222222222
123456789 × 27 = 333333333
123456789 × 36 = 444444444
123456789 × 45 = 555555555
123456789 × 54 = 666666666
123456789 × 63 = 777777777
123456789 × 72 = 888888888
123456789 × 81 = 999999999
222222222 × 9 = 1999999998
333333333 × 9 = 2999999997
444444444 × 9 = 3999999996
555555555 × 9 = 4999999995
666666666 × 9 = 5999999994
777777777 × 9 = 6999999993
888888888 × 9 = 7999999992
999999999 × 9 = 8999999991

I have no wish to over-emphasize arithmetic in our course of study. Any sixth grade teacher knows that she must cover much ground in teaching many other subjects. Occasionally correlation with arithmetic proves valuable. In science the study of the solar system has provided information and entertainment for pupils who have devised recreational type units. They prepare these in committees and small study groups beyond the minimum assignment. Our language arts activities include dramatizations. We have written and presented several dramatizations based upon arithmetic. In the social studies our units the Latin American countries, the British Isles, Canada, and Australia. Possibilities for recreational units involving arithmetic are obvious. Comparisons and contrasts relative to size in area, population, etc. always strengthen knowledge in both fields. I have had many such units worked out by pupils. Subject matter of the sixth, seventh, and eighth grades is especially good for incidental correlation of other subjects with arithmetic.

Drills and number games for fun can be used in lower grades, but recreational arithmetic begins to come into its own in grades four through eight, after basic fundamentals have been mastered. Of course, these same fundamentals need constant review.

Because sixth-graders love guessing games, I may start one out by saying, "Think of a number between 1 and 10, and I'll bet I can guess your number! Write it down, but don't let anyone see it. Now multiply your number by 3. Add 1. Multiply the result by 3. Add the number which you selected in the beginning and be ready to tell me your final answer. I'll promise to guess your original number if you have made no careless mistake!" By striking out the units digit in the final answer, the child's number will remain, but of course I don't tell that "secret" to the class. Until they discover it, they are amazed at my ability, and

they all clamor to have me guess their chosen numbers. I may add a few more difficult guessing tricks with two or three digits. When the secrets are exposed, the pupils all try the tricks on Mom, Dad, Big Brother, or Sister when they go home. It gives my sixth-graders a chance to "show-off" at home, perhaps to gain added prestige. Often, as Dr. Willerding has pointed out, children can establish, through recreational arithmetic, a new type of comradeship with the elder members of their families. This leads to an exchange of tricks and puzzles, which stimulates interest in mathematics. Eyes and ears are alerted for new tricks to try. I am always delighted when my pupils bring a trick worked out at home or spotted in a newspaper or magazine.

Guessing ages, birth dates, small change, pages in a book, etc. is sufficient fun to prove to an eleven year old that working with numbers is not all dreary business.

On the basis of personal experience, I feel impelled to reiterate that, going beyond the point of having fun, working with magic squares and cross number puzzles can actually strengthen mathematical ability. When I present a magic square to my class, I explain what one is and how it is constructed. I demonstrate with one simple 9-celled square and tell the class that there are seven other possible combinations. Within a few minutes many pupils come up with the other squares filled in; they beg for more patterns, larger and more difficult squares.

Cross-number puzzles serve as good review of basic processes. I teach area, perimeter, percents and decimals as applied to practical, every-day life, of course. But the necessary processes and formulae can be reviewed in cross-number puzzles. These are not usually included in text-books, and, while the text-book is important in an arithmetic program, materials not found between its pages seem to have more magnetism for the minds of sixth-graders. If the pupils like the puzzles

MAGIC SQUARE

4	9	2
3	5	7
8	1	6

I give them for review, they make puzzles of their own to exchange with classmates. None of these has been required, but I have been surprised at the number of puzzles turned in on a voluntary basis. My files are full of really good recreational materials which are the original creations of pupils.

When youngsters choose to use leisure time to "play" with numbers and when they beg to remain indoors at recess to work on number puzzles (which plea I do not grant in good weather!), I am sure that sixth-graders do like arithmetic. But of course the true test comes in evaluating the results of a program where recreational methods are used. Standard tests show the pupils' performance to be well above the norm. Each year the improvement in arithmetic achievement is gratifying. During this year the progressive growth in skills has been especially noteworthy. I must in no way imply that recreational arithmetic is solely responsible for a good achievement record. There are various contributing factors.

I recognize that many teaching procedures are necessary for a thoroughly rounded arithmetic program. Recreational devices are only among the many, but their value should not be overlooked. Some teachers will say that they have no time for fun in arithmetic. I think that they should allocate some time for it. I believe improvement in pupils' skills and attitudes will result.

Addition and Subtraction Situations

John Reckzeh
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In his article, "Which Way Arithmetic?" Professor Van Engen[1] implicitly defines an additive or addition situation as one in which groups are joined to form a single group. He specifically states that there are three additive situations as follows:

(1) To a group of known size, it is required to determine how large a group must be added to obtain a larger group of known size. 6 plus what number is 9 or $6+N=9$ is typical of this type.

(2) To a group whose size is not known, a group of known size is added and the size of the resulting group is known. The problem is to determine the size of the original group. What number plus 6 is 9 or $N+6=9$ is typical of this type.

(3) Two or more groups *of known size* are joined to form a single group. Addition determines the size of the single group.

Situation (3) above is the one that most people will recognize immediately as being additive. That is, any problem which produces a situation of type (3) will require addition for its solution.

Since (1) and (2) do involve the joining of two groups (one of which is unknown in size) to form a single group, these situations must be considered additive by the definition above. However, problems which produce either situation (1) or (2) must be solved by subtraction. Thus, the definition stated in the first sentence of this article produces the somewhat unexpected result of having problems which must be classified as additive but which require subtraction for their solution.

It is worth pointing out at this point that one of the major developments in the meaningful teaching of arithmetic is an approach to problem solving that is based on the nature of the groups of objects or things involved in the problem and how these groups are altered or redistributed in the course of the solution of the problem. Certainly it is no longer defensible to use the old fashioned method of saying that certain key words such as *more than*, *less than*, and *of* indicate the operations of addition, subtraction and multiplication, respectively.

An approach to problem solving that uses terms such as "additive situation" and "joining of groups" is certainly in accord with the best practices accepted to date in teaching arithmetic meaningfully. However, a terminology that requires that problems producing additive situations which must be solved by subtraction seems unfortunate. It should be recognized that in analyzing problem solving in arithmetic, the basic situations encountered must be accepted as they are found, but the names applied to these situations are arbitrary, with the exception that tradition has, in many instances, fixed certain names for certain situations.

The concept and term of "additive situation" are quite new and tradition has not had sufficient time to set the accepted meaning for this term as yet. As is pointed out in "Which Way Arithmetic?", arithmetic has been considered to be the processing of numbers and not the analysis of situations. It would certainly seem desirable to find a terminology that would have problems with additive situations solved by addition. It seems almost self-

[1] H. Van Engen, "Which Way Arithmetic?", The Arithmetic Teacher, December, 1955, p. 131.

evident that much less confusion would result in the learning stages if it were possible to choose a terminology that associated only addition with additive situations. IT IS POSSIBLE. The following definition of an additive situation makes it possible.

Redefining Additive Situations

AN ADDITIVE SITUATION is one in which two or more groups *of known size* are joined to form a single group where the size of the latter group is to be determined.

This is, of course, the situation (3) referred to in the beginning of this article. If this is accepted as *the* definition of an additive situation, then situations (1) and (2) are no longer additive situations even though joining does take place. They are not additive because the size of both of the groups being joined is not known.

Professor Van Engen states in his article, "Children are taught that when groups are joined the process of addition is used." If the children are taught that when groups *of known size* are joined, the process of addition is used, the problem of terminology will be much simplified and addition will be the only operation required for additive situations.

If the definition (3) is accepted as the only additive situation, the problem of how to classify and solve problems producing situations (1) and (2) is unresolved. Since subtraction is necessary for the solution of such problems, it would seem natural to classify such situations as subtractive or subtraction situations. However, in analyzing all problems which require subtraction for their solution, one discovers that no one description is sufficient. Indeed, some authors list four distinct subtraction situations. As has been pointed out, the situations that produce subtractions are as we find them, but to inspect these situations so carefully that subtle differences are over emphasized does not seem to be in the best interest of learning on an early elementary school level. While any mathematician and many others will readily distinguish between situations (1) and (2), there is a strong argument for emphasizing the similarities rather than the differences between these two situations. They both involve the joining of two groups, the size of one being unknown, to form a single group of known size, and they both require subtraction for solution. There seems to be little justification to present (1) and (2) as separate situations in the elementary school when they can be presented under the single situation just described. There is certainly no violation of any basic mathematical principle in this approach. Indeed, the algebraic statements of the typical problems cited, $6+N=9$ and $N+3=9$, would almost never be classed as different types of equations.

Defining Subtraction Situations

From the point of view of the elementary school, the basic subtraction concept is the *take away* concept, and the basic subtraction situation would then be defined as below.

THE BASIC SUBTRACTION SITUATION is that in which known part of a group of specified size is removed. The purpose of performing the subtraction is to determine the size of the remaining portion of the original group.

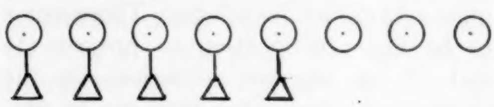
To present problem solving for problems which involve subtraction on the basis of four distinct subtraction situations becomes a rather formidable job. There can be no denying, however, that these situations do exist. For example, one very common subtraction situation which has not been mentioned as yet involves comparison.

The comparative subtraction situation occurs when two *distinct* groups of known size are compared to decide how much more or less one is than the other.

This situation occurs frequently in problems encountered in the elementary school. A sensible approach would seem

to be to emphasize the relationship of this situation to the basic one described previously. Consider the following problem.

Compare a group of eight circles with a group of five triangles to determine how many more circles there are than triangles. To solve this problem, arrange the circles and triangles as demonstrated:



With this arrangement, it can be seen that the problem can be solved by taking away from the original group of eight circles, a group of five circles (one for each triangle). In this manner, a problem producing a comparative subtraction situation is solved by reducing it to the basic subtraction situation.

Since problem solving in arithmetic is largely that of penetrating the verbal description of the problem to determine the situation involved, it is important that students be given the opportunity to interpret situations verbally. There are four common ways in which the subtraction just encountered might be described.

1. A group of five circles removed from a group of eight circles leaves a group of three circles.

2. The difference between a group of eight circles and a group of three circles is a group of three circles.

3. A group of five circles is three less than a group of eight circles.

4. A group of eight circles is three circles more than a group of five circles.

The first statement is a literal description of what actually happens in the basic subtraction situation as previously described.

The next three statements show how the process of subtraction is described by the commonly used terms, *difference*, *less than* and *more than*. No student can be said to have a mastery of subtraction from the verbal interpretive point of view if he cannot interpret subtraction in these four ways.

The fourth statement has an additional feature which is very important. It points out the important relationship that by adding the difference to the subtrahend, one obtains the minuend. This is essentially a statement of the inverse relationship between addition and subtraction. It is also the basis for the additive method of subtraction, to say nothing of the fact that it is introduced as a means of checking subtraction almost immediately after subtraction itself is introduced.

With this phase of the discussion completed, it is now possible to discuss the solutions of problems producing the "additive situations" (1) and (2) as discussed by Van Engen. Consider the following problem.

A boy wants a $50 bicycle but has only $20. How much more must he save?

A student noting that the joining of two groups is involved should also note that the size of only one of the groups to be joined is known so that it is not an addition situation. A student familiar with the basic subtraction situation and the four ways of interpreting it could solve the problem in the following steps:

a. A group of $20 removed from a group of $50 leaves a group of $30.

b. This means that $50 is $30 more than $20 and that therefore $30 more is required before the bicycle can be purchased.

The solution of a problem producing situation (2) is essentially the same. Only a change in the order of addition is required and most students do this naturally on the basis of experience.

While the opinions stated in this article are based on a very considerable amount of informal experimentation, there is little or no scientific evidence to back up these opinions. To my knowledge the same can be said for the opinions of Professor Van Engen. In fact the entire meaning theory, which is currently accepted by the bulk of

mathematics educators today, has an appallingly small amount of authentic research on which to base its claims. There is a great need for careful research to investigate many of the issues discussed here.

Summary

A summary of the discussion thus far should include the following major points:

1. Both this article and that of Van Engen advocate an approach to problem solving based on the recognition of the situation described by the problem. Both agree that the approach which attempts to connect key words with the processes of addition, subtraction, multiplication and division is old-fashioned and undesirable.

2. The major difference is one of terminology. This article advocates that a situation be called additive only when all problems producing this situation must be solved by addition. The same principle should apply to subtraction, multiplication and division situations.

3. This article proposes that there is only one additive or addition situation and that this occurs when two or more groups *of known size* are combined into a single group. The purpose of addition is to determine the size of the single group.

4. This article discusses the following subtraction situations:

   a. *The take away situation* which is called the basic subtraction situation. In this, part of a group is removed and the subtraction determines what is left.
   b. *The comparison situation.* In this two distinct groups are compared to determine how much more or less one is than the other.
   c. *The joining situation* where one of the joining groups is unknown in size, but the size of the final group is known. Subtraction determines the size of the unknown group.

5. This article advocates the use of 4a (above) as the basic subtraction situation and that the other situations be interpreted in terms of 4a as described above. It should be noted that both of the "additive situations" (1) and (2) are included in situation 4c. This is in line with the principle that the similarities between the situations should be emphasized as much as possible, rather than the differences, during the early stages of learning.

6. Given the subtraction $x-y=d$, the student should be able to interpret it verbally in the following four ways:

   a. A group of y items removed from a group x items leaves a group of d items.
   b. The difference between a group of x items and a group of y items is a group of d items.
   c. A group of y items is a group of d items less than a group of x items.
   d. A group of x items is a group of d items more than a group of y items.

The question of timing, or over what period of time should these ideas be introduced, has not been discussed. This will largely be determined by the text used in most situations.

A discussion of multiplication and division situations will follow at a later date.

Editor's Note. Both Messrs. Reckzeh and Van Engen have discussed basic ideas which must be considered by those who are concerned with the direction of arithmetic programs. Both desire the understanding of relationships in a situation as a key to the process involved. Mr. Reckzeh points out the danger in too much refinement for children at the early stages of learning. There is however a need for teachers to think clearly about situations and their mathematical content. We must not lose sight of our basic aim and that is to teach pupils (a) to recognize a situation that is mathematical, (b) to understand the mathematical relationships involved, (c) to think in terms of the needed mathematical elements, and (d) to perform satisfactorily in the judgments and computations needed for a valid conclusion.

Inconsistencies in the Teaching of Arithmetic

Part II

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IN THE MARCH ISSUE, thirty test items were offered and the reader was challenged not only to give the correct answer but also to give a rational explanation which children could understand. The author's explanations are given below.

The criteria for determining the accurateness of the statements or procedures in teaching elementary arithmetic are based on the following aims:

1. Make arithmetic meaningful
2. Build a logical structure

Language is an important factor in the accomplishment of these aims. Therefore statements should be clear and complete so that pupils will not get incorrect concepts.

The incorrect concepts discussed in this article are the results of the author's experiences with two distinct groups. The first group consists of pupils in grades three through six with whom the author served as a teacher or a consultant. The second group consists of elementary grade teachers with teaching experience ranging from one to twenty-five years. The author's contact with this group came through workshops, in-service training, and graduate classes in the teaching of elementary arithmetic.

Many of the incorrect concepts of pupils are due to the poor or evasive language of teachers and incorrect concepts of teachers. Often a teacher knows the correct answer or what she means but the language used produces only mechanical action on the part of the pupil. If arithmetic is to be a logical structure and is to be meaningful to the pupils each new thought should be developed on the level of experience of the pupils from the concepts already understood.

Each item in the test in part I was chosen from a large group of questions given to approximately 100 experienced elementary grade teachers from both small and large school systems. Each of the above questions will be repeated with the correct answer and a discussion of the concepts involved. Following each discussion is the percent of the experienced teachers who gave the correct answer. In some cases we are not so much concerned with the correctness of the answer as with the rationalization of the answer at elementary grade levels in terms of concepts and principles already presented, that is, build a logical structure!

Test Items and Explanations

1. __F__ $5+2\times5=35$

(1) 2×5 means $5+5$ hence $5+2\times5=5+5+5$.

(2) One of the fundamental principles taught in early grades is $A+B=B+A$, that is, addition is commutative. When the "+" sign occurs with no signs of grouping the order of addition may be reversed hence $5+2\times5=2\times5+5$. However, $(5+2)\times5$ does equal 35.

(3) If we put a five dollar bill (one five) in the bank on Monday and two five dollar bills (two fives) in the bank on Tuesday obviously we have deposited 15 dollars or $\$5+2\times\$5=\$15$.

(4) Certainly the reader will accept the statement that $5+2A$ is *not* equal $7A$. But $2A$ means 2 times A hence $5+2\times A$ is not equal to $7A$. If A is 5 then $5+2\times5$ cannot equal 35.

(5) $5+2\times5$ may be thought of as 5 ones+2 fives and since we cannot

add unlike things we cannot get 7 fives or 35.

Therefore if we are to build a logical structure in arithmetic we cannot teach them to do computation from left to right but in a statement containing both multiplication and addition we must teach them to multiply first then add unless signified to do otherwise.

Only 10% of the experienced teachers answered this question correctly.

2. F 0 times 5=0.

Zero as a multiplier cannot be rationalized. By definition multiplication is a short method of finding, without adding, the sum of a given number of like addends by using certain established facts. In other words 3 times 5 means $5+5+5$. The five is the multiplicand and the 3 is the multiplier telling how many times the multiplicand is to be used as an addend. Try and explain 0 times five by this definition. Some would say take 5 zero times and add and get zero. But how can you add any numbers (except zeros) and still get zero! This does not mean that zero cannot occur in the multiplier for 104×236 means an absence of a multiplier in ten's place.

Only 3% of the experienced teachers answered this question correctly.

3. F A whole number may be multiplied by ten by adding a zero to the number.

In the primary grades we teach zero added to a number gives the same number in the sum, yet it is a common practice of teachers in the latter part of third grade to teach "10 times 3 can be obtained by adding a zero to the 3 getting 30." This is pure mechanical. It *is not consistent* with earlier teaching. That is, adding zero to a number means one thing in the second grade and another in the third grade. Although many teachers use this statement, the statement itself rarely occurs in print.

$$\begin{array}{r} 3 \\ +0 \\ \hline 3 \end{array}$$

Only 3% of the experienced teachers answered this question correctly.

4. F $2\frac{1}{3}=\frac{3\times 2+1}{3}=\frac{7}{3}$.

The above computation is a common method of changing a mixed number to an improper fraction. It is mechanical, the result of a rule, and cannot be rationalized. The correct statement should be

$$2\frac{1}{3}=\frac{2\times 3+1}{3}=\frac{7}{3},$$

for we change 2 wholes to thirds and there are three thirds in each whole, hence there are 2×3 thirds in the 2 wholes.

Only 5% of the experienced teachers answered this correctly.

5. F A two-inch square contains two square inches.

A two-inch square contains four square inches. This incorrect idea often occurs from the definition of an area of one square inch. A square each of whose sides is one inch is defined to be a unit of area of one square inch. This does *not* say that an area of one square inch must be in the shape of a square. The author has found many teachers will often accept as true the converse of a true statement. Another common example is: "*If* the excess of nines in the answer of an addition exercise equals the excess of nines in the sum of the excesses of the addends *then* the computation is correct." This is not a correct statement although it is the converse of a true statement: "*If* the computation is correct *then* the excess of nines in the answer of an addition exercise will equal the excess of nines in the sum of the excesses of the addends."

About 65% of the experienced teachers answered this question correctly.

6. F Five times the sum of seven and four equals the sum of five times seven and four.

In symbols this means

$$5\times(7+4)=5\times7+4.$$

It is not correct. The language used with elementary pupils is very important. One should also be critical, in a constructive way, of the language used by pupils, especially when two or more ideas may result from a statement.

Approximately 60% of the experienced teachers answered this correctly.

7. F John has eight dollars and Henry has four dollars. Compare the value of John's money to the value of Henry's money. Answer: *It is two times greater.*

The answer implies that $8 is two times greater than $4. This means that $8 is two times $4 greater than $4. This is incorrect. The answer should be similar to one of the following: "It is twice as great" or "John has twice as much as Henry."

Forty per cent of the experienced teachers answered this question correctly.

8. F Addition is a quick way of counting.

This is a loose definition and may lead to various incorrect concepts. The statement implies counting may be performed by the addition process. It is true that situations involving addition may be solved by counting but the converse is not always true.

The concept of addition involves two or more distinct numbers or groups while counting is setting up a one-to-one correspondence between something to be counted and a set of counters. "Addition is the process of finding the sum of two or more numbers without using the counting process" or "Addition is the process of finding without counting the total number in two or more groups of things."

About 40% of the experienced teachers answered this correctly.

9. F When subtracting take the smaller number from the larger number.

This is another loose statement and may lead to such work exhibited at the right in which the pupil always subtracts the smaller digit from the larger digit.

$$\begin{array}{r} 3\ 2\ 4 \\ -1\ 7\ 8 \\ \hline 2\ 5\ 4 \end{array}$$

Approximately 30% of the experienced teachers answered this question correctly.

10. F Multiply means to increase.

The product is greater than the multplicand only when the multiplier is greater than one.

About 60% of the experienced teachers appeared to understand this situation.

11. F Multiplication is a short way of adding.

This is another loose statement from which pupils may get ambiguous concepts. The author is reminded of the pupil that said, "Let's find the answer to 619+732+586+174 by multiplication because it is quicker."

A definition of multiplication should be descriptive and give the readers or listeners some idea how this particular process differs from other processes. A good definition of the multiplication process is given in the discussion of question 2.

Twenty per cent of the experienced teachers answered this question correctly.

12. F 2 feet$\times$3 feet = 6 square feet.

This statement cannot be rationalized to the elementary pupils. The definition of multiplication excludes the use of a denominate number for a multiplier. In a multiplication exercise the product has the same name as the multiplicand. The statement should read "2$\times$3 sq. ft. = 6 sq. ft."

Only 20% of the experienced teachers answered this question correctly.

13. F The symbol 9 has a larger value than the symbol 1.

This is not always true because the value of a symbol depends not only on its form value but also on its place value. In the number 519 the "1" has a larger value than the "9." Pupils that understand our number system may be led to make a general statement, "In a given number a symbol (except 0) always has a larger value than a symbol on its right."

Eighty-five per cent of the teachers answered this question correctly.

14. __F__ 5 feet$\times$6=30 feet is read as "five feet times six equals thirty feet."

The multiplier is 5 feet, hence it is incorrect. The correct statement is "5 feet multiplied by six equals thirty feet."

Thirty per cent of the experienced teachers had this answer correct.

15. __F__ A number may be multiplied by ten by annexing a zero on the right.

Many sixth grade pupils know this is not correct because annexing a zero on the right of 2.3 does not multiply 2.3 by ten. However a whole number may be multiplied by ten by annexing a zero on the right.

Thirty per cent of the experienced teachers answered this question correctly.

16. __F__ Mary bought 5 pounds of steak that cost 80 cents a pound. Find the total cost of the steak (solution at right).

$$\begin{array}{r} 80¢ \\ \times 5 \text{ lbs.} \\ \hline 400¢ \end{array}$$

This is not a correct solution since the multiplier must be an abstract number, however the answer is correct.

Seventy per cent of the experienced teachers answered this question correctly.

17. __F__ 5$\times$.25=\$1.25

The correct answer is 1.25. The answer has the name "dollars" only if the multiplicand has the name dollars.

Ninety per cent of the experienced teachers had this answer correct.

18. __F__ Increasing the divisor and dividend by the same number does not change the value of the quotient.

Two possible sources of this incorrect concept are:

(1) To a few people "increase" in certain situations indicates multiplication.

(2) Some people learn only the conclusion of a statement without attaching the conclusion to its corresponding hypothesis. Hence, "doing the same thing to divisor and dividend does not change the quotient," when the correct statement involves *multiplying* (or *dividing*) the divisor and dividend by the same number.

Seventy per cent of the experienced teachers answered this question correctly.

19. __F__ Division is a short way of subtracting.

This is another poorly worded statement. Division is a process of repeated subtractions. The divisor is used as a subtrahend until the remainder is less than the divisor.

Only 20% of the experienced teachers answered this correctly.

20. __F__ 8 square feet$\div$2 feet=4 feet.

Either the divisor or quotient must be an abstract number and the other must have the same name as the dividend. If we use the "repeated subtraction" definition of division we could not subtract 2 feet from 8 square ft. If we use the "equal groups" concept of division then either 8 square feet is to be separated into 2 equal groups with 4 square feet in a group or 8 square feet is to be separated into equal groups with 2 square feet in a group giving 4 equal groups. If we refer to the division process as the inverse of the multiplication

progress then either the divisor or the quotient must be an abstract number.

Twenty per cent of the experienced teachers answered this question correctly.

21. __F__ If 12 dollars is divided into two groups each group is called one-half.

Often we fail to emphasize the idea of "equal groups" in division exercises or in situations involving fractions.

Eighty per cent of the experienced teachers answered this question correctly.

22. __F__ $8 \div 0 = 0$.

This cannot be true since 0×0 does not equal 8. One can easily show the pupils in the higher grades the answer to such an exercise is not zero by using the following sequence:

$$\frac{8}{1}=8; \quad \frac{8}{.1}=80; \quad \frac{8}{.01}=800;$$

$$\frac{8}{.001}=8000; \quad \cdots; \quad \frac{8}{.000001}=8{,}000{,}000;$$

etc., hence as the divisor gets nearer and nearer to zero the quotients gets larger and larger, therefore the quotient cannot be zero.

Only 10% of the experienced teachers answered this question correctly.

23. __F__ $18 \div 2 \times 5 = 45$.

This exercise is difficult to explain to elementary grade pupils but it is rather unimportant since it rarely occurs in this form in the elementary grades. Some authors say we should begin our operations on the left in such situations as this. Yet these same authors give such exercises as $18 \div 2a = 9 \div a$ or $9/a$. This is inconsistent since $18 \div 2a$ is no different from $18 \div 2 \times 5$ where $a = 5$. Since more than one answer may occur here we ought to teach pupils to perform operations in the order; multiply, divide, then add and subtract, unless otherwise indicated by grouping signs. Hence, $18 \div 2 \times 5 = 1.8$ and $(18 \div 2) \times 5 = 45$.

Only 10% of the experienced teachers answered this question correctly.

23. __F__ $\frac{3}{8}$ divided into $\frac{1}{2} = \frac{3}{\underset{4}{\cancel{8}}} \times \frac{\overset{1}{\cancel{2}}}{1} = \frac{3}{4}$

This error probably results from the fact that we have various ways of stating a division exercise. Too often we stress only "invert and multiply." We ought not emphasize form only but which is the number to be divided (dividend) and which is the number to divide by (divisor).

Eighty per cent of the experienced teachers answered this question correctly.

25. __F__ $5 \times .25¢ = \$1.25$.

The name of the multiplicand is cents, hence the answer is 1.25 cents.

Eighty per cent of the experienced teachers answered this question correctly.

26. __F__ In the number $2\frac{1}{3}$ the fraction $\frac{1}{3}$ means $\frac{1}{3}$ of 2.

The $\frac{1}{3}$ means $\frac{1}{3}$ of the value of the preceding place value. Teachers often fail to emphasize this principle.

Eighty-five per cent of the experienced teachers answered this question correctly.

27. __F__ 3 will divide into 21 an even number of times.

Three will divide into twenty-one seven times and seven is an odd number. This error is probably due to the common use of the word "evenly"; that is 3 will divide into 21 evenly; meaning, with remainder zero. This may be trivial but certainly the language is not correct.

Not one of the experienced teachers answered this correctly.

28. __F__ $.0\frac{1}{3}$ means $\frac{1}{3}$.

By the principle stated in the discussion of question 26 the correct answer is $\frac{1}{3}$ of $\frac{1}{10}$ or $\frac{1}{3}$ tenth.

Seventy-five per cent of the experienced teachers answered this correctly.

29. __F__ $.0\frac{1}{3}$ means $\frac{1}{3}$ of $\frac{1}{100}$ or $\frac{1}{3}$ hundredth.

The correct answer is $\frac{1}{3}$ of $\frac{1}{10}$ or $\frac{1}{3}$ tenth.

Only 30% of the experienced teachers answered this question correctly. Obviously the great difference in the correct answers to 28 and 29 is that many thought $.0\frac{1}{3}$ means $\frac{1}{3}$ of $\frac{1}{100}$ or $\frac{1}{3}$ hundredth, that is $\frac{1}{3}$ appears in hundredth's place.

30. __F__ In the subtraction exercise at the right 7 tens cannot be taken from 2 tens so we take (or borrow) one hundred ones from the three hundreds and change them to tens.

$$\begin{array}{r} 3\ 2\ 7 \\ -1\ 7\ 4 \\ \hline 1\ 5\ 3 \end{array}$$

Summary

It is possible that poor achievement of pupils in the fundamental processes is due to a lack of thorough understanding of our number system. In our number system we have place value. It takes ten units of one place to make one unit of the next higher place. The 3 in hundred's place means three hundreds. But how do pupils think of those three hundreds? Are there three hundred ones? Or three groups with one hundred ones in each group? Or three groups with one hundred in each group and each hundred is composed of ten tens, and each ten is composed of ten ones? The latter is the exact concept. Too many pupils think of a unit of a certain order in terms of ones rather than units of the next lower order. Thus a unit of hundreds is thought of as being one hundred ones rather than ten tens. It is true that a unit of hundreds is equivalent to one hundred ones. But to change a unit of hundreds to ones it must be first changed to ten tens.

If the word "ones" is omitted the given statement in question 30 is correct.

Very few of the experienced teachers understood the implication in this question.

It is reasonable to assume that teachers who do not have a thorough understanding of the concepts and principles which underlie the processes and procedures of arithmetic cannot lead pupils to develop arithmetic as a logical structure. Without this understanding the language of teachers will not be descriptive and exact and the results will in general be "mechanical" arithmetic.

Editor's Note. Do you agree with Dr. Dwight's explanations? Are they on a level that an elementary school pupil can understand? Are there certain parts of arithmetic that a pupil should be asked to accept without full explanation? Should a pupil realize that some things, like certain rules of operation, are agreements which must be observed to avoid confusion in a logical structure? Should we differentiate on the basis of pupils' abilities the amount and nature of understanding expected of them? Is it fair to assume that teachers should understand the mathematical backgrounds of arithmetic so that they will not lead pupils into misunderstandings? How will teachers gain such a background when their training colleges do not offer it?

California Conference

California's conference for teachers of mathematics will be held on the Los Angeles Campus of the University of California June 20–July 3, 1956. The conference will deal particularly with the unifying concepts of mathematics from grades one through twelve. Other study groups will be concerned with the curriculum and with counseling mathematics students. There will be opportunity to enroll in mathematics laboratories in which teachers will learn to make and to use many visual aids. Interesting talks by outstanding specialists will be presented. For information write to Professor Clifford Bell, University of California, Los Angeles 24, Calif.

Evaluation by Observation—Grade 3

Alice P. Thomson
New York City Public Schools

At the end of grade three, it is possible to evaluate the pupil's knowledge of number facts and processes and of some understandings through written tests. However, there are some mathematical understandings that are better evaluated through observation. While children are engaging in their daily work and experiences you will find many opportunities for observation. If such opportunities do not occur, they may be contrived. Among the understandings that may be observed and evaluated by teachers in grade three are such as the following:

Concerning Measurement

Money
Shape
Size and Distance
Bulk and Liquid
Weight
Temperature

Concerning Fractional Parts of

Bulk
Liquid
Material
Whole Objects
Numbers of Things

In teaching measurement, it is advisable to use familiar, non-standard units of measure before you use standard measuring units. For example, you use the familiar household cup before you use the standard measuring cup; the common teaspoon before the standard measuring spoon; jars of different kinds before a pint or quart container. After some experience with this informal type of measurement, you can then present a standard unit of measure. In this way, your children will have an opportunity to think about measurement and to find out that there are all kinds of measures, as well as standard measures.

Money and Counting

By the time children reach the third grade you will find that most of them have used real money in out of school activities. They know that it is needed to buy candy, ice cream, food, toys, etc. Some of them have already learned that the same thing can be bought for five pennies or a nickel. The third grade teacher makes sure, however, that all pupils are able to identify the coins and know the equivalents for them in pennies and other coins. She can observe this as children engage in experiences, real or contrived. Such experiences will be a basis for meaningful understanding for some children, and will be practice and reenforcement for others.

Money is very useful to the teacher in helping children count on more advanced levels than by ones. At first they count only pennies—starting with a number they already know, and then count by twos or by other groups if they can. When they use nickels, dimes, quarters and half dollars too, they have opportunities for all kinds of counting.

Money is also useful in teaching the meaning of addition and subtraction in the third grade. One of the most important areas in third year mathematics is the development of the understanding of the place value idea of numbers. This is most easily approached through work with money. Our money is based on our decimal number system, a tens system. This makes it easy to use money to teach the place value idea of numbers. Children like to use money, and it helps them understand our number system.

The Observation

The pupil's ability to recognize and use money can be observed during these and other activities and experiences:

Playing store
Buying and selling cookies
Collecting for milk, lunch, Red Cross, etc.
Planning or paying for trips

It seems unlikely that there will be any third year pupils who have had no experience with money in school or out of school. Since they have such a varied background you will find it helpful to use the suggested sequences beginning with first year money concepts, as you plan to help your children use and understand money and counting.

If, because of absence or other reasons you have some pupils whose competency you have not been able to note during their experiences, you may test them individually or in small groups, and proceed as follows:

Have available more than 100 pennies, 20 nickels, 10 dimes, 4 quarters, 2 half dollars and a 1 dollar bill.

1. Count pennies by ones, twos, etc.
2. Ask the pupil to select each coin and the bill by name, and tell its equivalent in pennies. Let him count the equivalent in pennies for several of these coins.
3. Let him tell for some and count out for others, the following equivalents:
   a. Nickels in a dime
   b. Nickels in a quarter
   c. Various combinations of pennies, nickels and dimes in a quarter
   d. Number of dimes in a half dollar
   e. Number of nickels in a half dollar
   f. Number of quarters in a half dollar
   g. Various combinations of coins in a half dollar
   h. Number of dimes in a dollar
   i. Number of quarters in a dollar
   j. Number of half dollars in a dollar
   k. Number of nickels in a dollar
   l. Various combinations of coins in a dollar
4. Let him make change for any amount (from any coin or the dollar bill) in an actual situation, e.g.

   Tom pays John a quarter (half dollar or dollar) for milk. Milk costs 16¢. John says, "sixteen cents," and giving Tom 4¢, says, "Twenty cents," then giving Tom a nickel, says, Twenty-five cents." Tom checks by counting the change he received, in the same manner (additive method).
5. Let him approximate the amount of money of any group of exposed coins whose total does not exceed 1 dollar. (So that the pupil may arrive at a close approximation, do not use too many coins.)
6. Let him check his approximations by counting the coins. Look for most efficient method of handling the coins, i.e., selecting coins of greater value first, e.g., half dollar, quarter, dime, nickel, pennies. The pupil should say, "50—75—85—90—93."
7. Let him count a group of coins beginning with any coin that is a multiple of 5. Place on display table a quarter, then 2 dimes, then three nickels and then 5 pennies. The pupil should say, "25—35—45—50—55—60—65."(He may be able to count the 2 dimes or the three nickels as one.)
8. Let him count a group of coins beginning with any coin that is not a multiple of 5. Place on display table 3 pennies, then 1 quarter, then a dime and then a nickel. The pupil should say, "3—28—38—43."

Shape

Since children in the first three years are too immature to understand the meaning of shapes in geometric terms, we ask only that they *recognize* objects which have been cut into or that have surfaces with the following shapes:

Circle
Square (differentiate from a box and a cube)
Rectangle
Triangle

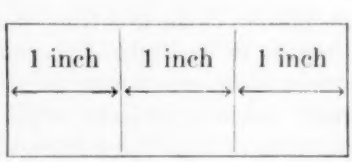
Size and Distance

Pupils develop the concept of measurement by measuring one familiar object with another familiar object, e.g., "How many times will this book fit across your desk?" "How many times will this pencil fit across this shelf?" It is important, too, that children be asked, before they actually measure, to *estimate* the number of times one object is contained in another. They need much practice in estimating size and distance.

After pupils have had many experiences in measuring with a non-standard unit, introduce the idea of *foot*. Relate it to the human foot. Let children with different sized feet measure the twenty foot distance needed for the eye chart. This will show them the need for a standard foot.

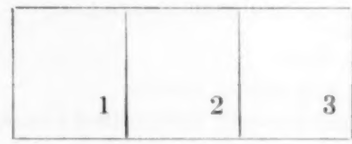
When a foot is presented, it is best that it be an unmarked, accurately measured foot, i.e., one made of oaktag or cardboard. Just a few trial measurements of objects with this unit will prove the need for a smaller unit of measure—the *inch*. Tell the children that *inch* was the name given to this length. Give each child a piece of cardboard one inch long. Let children try to measure objects with it. Awkwardness in handling it will quickly become apparent. Using cardboard inches, let the children develop the meaning of their own rulers. Six inches or less is sufficient for this development.

Have the pupils label one side, marking spaces with their one inch measures this way:



etc.

On the other side, they can then make marks like those on a regular ruler, this way:



etc.

More mature children can indicate the half inch with smaller markings.

Give the children practice in measuring many small objects, e.g., books, jewelry boxes, small paper, etc., with rulers they have made. Often, ask them to estimate the length before they measure.

Bulk and Liquid

Start with containers with which children are familiar. Let the children use a teacup or jar to fill or empty the aquarium or vase or pail. Let them count the number of cupfuls or jarfuls that are *contained* in the larger vessel. After some practice, let them try to *estimate* the number of cupfuls or jarfuls that will be needed. As children use containers of different sizes, they will see relationships among them. For example, an ordinary teaspoon and tablespoon can be used to develop equivalents. i.e., 3 teaspoons will contain the same amount of sugar as 1 tablespoon.

After many experiences with non-standard units of measure, present the standard units, i.e., cup, pint, quart, and standard measuring spoon. Practice in using standard measures will lead to the discovery of relationships among them.

Weight

In order to make weight meaningful, balance should be understood. Start with non-standard units of weight. Make a small see-saw. Let the children make it balance. Let them assist in making a balance scale. Use it to balance one object with one or more objects of the same weight, e.g., a block and discs, a book and blocks. After many, many such expe-

riences, introduce the standard measure of weight—*the pound*. Remember, a pound must be lifted to be understood, not looked at.

Standard weights are frequently unobtainable. Many substitutions may be made, e.g., a pound of coffee, two pounds of sugar. Present the pound weight, the two pound weight, etc. (Use standard weight or substitute.) Let the pupils hold the weights and compare them. They must *feel* and *sense* the difference between one pound and more than one pound to understand *weight*.

Temperature

In the beginning, the teacher helps children become conscious of heat and cold by asking them to touch things or to put their hands into pans of water of different temperatures. Most third grade children are able to compare temperatures, as warmer, cooler, etc., as well as hot or cold.

After experiences with comparing temperatures of water or objects, introduce the thermometer. Have the children observe the liquid in the tube and find the number nearest to the place to which the liquid has risen. Place the end of the thermometer in a dish of warm water. Can they see the liquid "grow?" Tell them that when the liquid gets warmer it "expands." At this point some teachers will want to help children perform other science experiments showing expansion of glass or metal. They will probably say, "Look at it *go up*." Then, do just the opposite with a dish of cold water. They will probably say, "See it *get smaller*," "*go down*." You can use the word "contract," and explain that when the liquid gets colder, it *contracts*. Give them many opportunities to see the mercury rise and fall.

The reading of the thermometer is really a reading skill. We do take advantage, however, of the numbers involved to use the reading of the thermometer for mathematical computation.

Fractional Parts

Even the youngest school children use fraction words to denote parts of things. They speak of half an apple, half a stick of candy, etc. Usually, they mean a piece, not a part in relation to a whole.

The work of the first three years is to develop the understanding of the relationships of parts to a whole, and of parts to one another. This must be developed with regard to a single object as well as to a group of objects, i.e., one half of an apple is one of two equal parts of one whole apple and one half of a group of 8 apples is one of two equal parts of that group.

In the first three years, the pupils show their understanding of fractions by demonstrating with real objects, and by telling in their own words, what they are doing. For that reason, the learnings in fractions are best tested by observation. Real things, like apples, pies, cakes, etc., are used in experiences. For the purpose of testing, circular cardboard discs may be used. Circular things are better for this purpose than other shapes, since it is easier for young children to recognize the fractional parts, when they are parts of circular wholes. Pieces of paper or a string are not appropriate as they seem to become smaller *whole* sheets of paper or string when cut into halves.

1. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$ *of a Whole Object*

In the beginning the pupils cut or fold many things of various sizes into halves. The pupils keep one whole circle intact for purposes of superimposing. After the children are thoroughly familiar with halves and their meaning, fourths are introduced in relation to the whole, and to halves. Finally, thirds are introduced. Let the pupils compare halves, fourths and thirds of like objects.

2. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$ *of Bulk or Material*

In the beginning the pupils will divide clay, sand, beans, etc., into parts—first

halves, then fourths and much later into thirds. The relationship of one part to another should be discovered. If you use paper or cloth, fold rather than cut, so that the parts of the whole will be discernible.

3. *Adding and Subtracting Halves and Fourths*

In the beginning pupils are really adding and subtracting halves and fourths as they relate them to the whole and to one another. Give them many opportunities to solve problems involving fractional parts within two wholes.

4. *Finding Fractional Parts of*

$\frac{1}{2}$ of even numbers of things through 20
$\frac{1}{2}$ of odd numbers of things through 10 (the odd 1 is left over, *not* broken in half)
$\frac{1}{4}$ of 8, 12, 16 and 20 things
$\frac{1}{3}$ of 6, 9, 12, 15, and 18 things.

As stated in the beginning, the work with fractions in the first three years is done entirely with objects. The symbol for fractions is not used by you or the children. For that reason, when finding fractional parts of numbers, objects must always be used, e.g., to find $\frac{1}{2}$ of 8, you must present either 8 pencils or 8 books, etc.

BOOK REVIEW

Explorations in Arithmetic, by Lowry W. Harding, 88 pages, $8\frac{1}{2}$ by 11 in., ring binder, Wm. C. Brown Co., Dubuque, Iowa, 1955. Price $3.00.

In this book Professor Harding of Ohio State University has bound together the sheets of materials he uses in a course in arithmetic for teachers. Many teachers of such a course prefer to make their own outlines and specify the course content. Mr. Harding's course seems to be more of a methods course than one of subject matter. He treats such topics as "What Do You Know About Teaching Arithmetic?", "Inventory of Professional Understandings in Arithmetic," "An Annotated Bibliography on Arithmetic and Arithmetic Education," Fostering Discovery With Children," "How Our Bank Helps Us," and "Self-Evaluation of Qualifications for Teaching Arithmetic."

This is a book that will be good for teachers to use in study groups and for those who are educating our future teachers. It is expected that elsewhere in the Ohio State University program of teacher education the students will have studied backgrounds of mathematics so as to assure a reasonable competence in the depth of content desirable for a teacher. The reviewer notes that Mr. Harding found in a survey of teachers who were trained in seven different institutions that teachers did not feel very secure in the training they had received in arithmetic. The background for teaching this area needs very serious consideration in training programs. Courses in Education do not provide the needed competence.

BEN A. SUELTZ

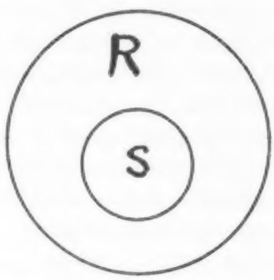
(*Continued from page 84*)

EDITOR'S NOTE. Dr. Wrightstone tells how the New York City group developed their tests for grades one and two. These tests are pictorially presented. It is significant that the actual test items were constructed by teachers who are working with the children. If we believe in the importance of concept development at the lower grade levels we should try to measure the aspects of arithmetic we are actually trying to teach. It is heartening to note that more and more tests, including standardized tests, are now attempting to measure concepts and understandings as well as the traditional computations and problems. While formally presented tests give a good deal of information about a pupil and a grade, it is still necessary to use a teacher's personal appraisal of some things such as depth of understanding, resourcefulness, and insight. But to do this reliably, a teacher must be alert and discerning in working with her pupils.

Enlarging Number Systems

CHARLES BRUMFIEL
Ball State Teachers College, Muncie, Ind.

WHENEVER WE ENLARGE a number system S we employ one of two methods. Each method has the same ultimate aim. The difference lies wholly in the paths we follow to obtain our heart's desire. In each case we end up with a system R which contains the system S, symbolically



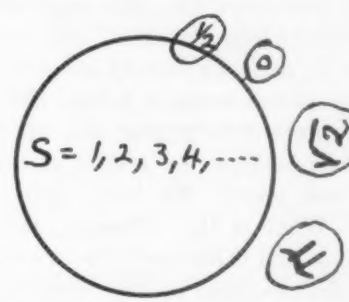
The usual procedure in passing from S to R consists of recognizing a computational situation which can not be satisfactorily handled using the elements of S. *If* it is clear that it may be useful to have "numbers" with which these "impossible calculations" may be effected, then *if* we have sufficient ingenuity we may be able to create such new numbers by:

(1) Naming them, i.e. assigning *written* and *verbal symbols* to them.
(2) Explaining how we use these new things in computation.

The *naming* is easy. It can be done in a moment. But the *explaining* is hard. Indeed the explaining may well extend over several years of schooling. But strictly speaking *a new number is not completely defined* for a particular individual *until he has a complete understanding of how it is used.*

We might call this method an extension by *adjunction*. Picturesquely we could describe it thusly:

We pick up these new numbers and hurl them at the number system S hoping desperately that when they strike they will stick. Unfortunately S has existence only in the mind and it is well known that the brain exudes an oily substance which encases the set S and makes it very difficult for these new numbers to attach themselves.



In the illustration we see that $\frac{1}{2}$ is firmly attached, zero is hanging by a thread, while $\sqrt{2}$ and π have bounced off and are lying on their backs kicking feebly.

How shall we handle the difficult part of our definition? How shall we explain the rules for computing with these new numbers? If our students are mathematically mature, we may be able to present only the formal definitions. Later we justify these formal rules by showing the magnificent results which are obtainable. Let us look at two such *precise, formally correct* definitions.

A. We assume familiarity with the natural numbers 1, 2, 3 $\cdots$ and the counting process. We denote the successor of the natural number x by x'. That is, $1'=2$, $3'=4$, $1000'=1001$ etc. We *define* addition of *two* natural numbers, indicated by $x+y$:

(1) $\qquad x+1=x'$

(2) $\qquad x+y'=(x+y)'.$

B. We *define* the product xy of two natural numbers:

(1) $$x \cdot 1 = x$$

(2) $$x \cdot y' = (x \cdot y) + x$$

The trouble with definitions A and B is that we have to teach addition and multiplication before our students are mature enough to understand these formal rules. *We have no choice! We can not define precisely. We must content ourselves for the moment with describing only.*

And so we replace "A" by some statement like: "*addition is finding the total number of things in two or more groups.*" Even a statement like this may well be long withheld from the child. Much teaching can be done by simply pointing and emitting an encouraging grunt. But it is most important to realize that we have *not defined addition* by such a statement as the one above. We have merely described. What is the difference between *description* and *definition*? *The description usually fails to apply in some situation and must be supplemented by further descriptions.* If we are excellent describers we are usually able to come up with new descriptions in such a way that eventually our students actually intuitively piece together a *definition* from our *descriptions*.

As examples of how our descriptions fail us, if we describe 4×3 as the "sum of four threes" then even by analogy 1×3 has *not* been defined. Further *description* is necessary. Of course if we had said "the sum of three fours" we would have been all right. But then, how about 3×1 and 1×1?

If we *describe* $12\div 4$ as the number of eggs in one sack if we distribute 12 eggs among four sacks, then we must think up another tale when we come to $2\div \frac{1}{2}$.

If we *describe* $\frac{1}{2}\times\frac{1}{3}$ as one half *of* one-third we will have to wriggle a little bit to explain $\frac{8}{5}\times\frac{2}{3}$ or worse yet $3\frac{1}{2}\times 2\frac{1}{4}$.

If we *describe* $3\frac{1}{2}\times 2\frac{1}{4}$ as finding the area of a plot $3\frac{1}{2}$ rods by $2\frac{1}{4}$ rods we will have to meditate anew when we wish to explain how multiplication is related to cutting off $\frac{2}{3}$ of one-fourth of a pie.

Probably the best teaching procedure is to set up many problem situations for the child in which he must secure answers by "reasoning" alone. That is, formulate problems which are "naturally" solved later on by the application of formal calculations not yet known to the child. For example, before multiplication and division of fractions have been *described* let us ask such questions as

1) If I cut a fourth of a pie into two equal parts what do I call each piece?
2) If I have two and one-half dollars and give each child a dime, how many may be the recipients of my generosity? (It would seem best to reword this for the second grade.)
2) If a rectangular piece of land is $2\frac{1}{2}$ miles wide by 4 miles long, how many square miles does it contain?
4) How many tenths of a mile must you travel in order to go one-fourth of a mile?

After skillful teachers have asked enough questions of this sort a solid foundation may (perhaps) have been laid for the introduction of a new, formal calculation. If this be multiplication of fractions and the child has become well versed in area concepts, then *describe* $\frac{1}{2}\times\frac{2}{3}$ in terms of finding area, i.e. remark that by this we mean that we are to find the area of a rectangular plot $\frac{1}{2}$ foot *by* $\frac{1}{3}$ foot. (The bright pupils will probably convert to inches.) If the child and his teacher like the same kind of pie, *describe* by saying that $\frac{1}{2}\times\frac{1}{3}$ means to take one-half *of* one third of a pie. Here the $\frac{1}{3}$ is pie and the $\frac{1}{2}$ is student with a big sharp knife and sealed instructions. If the child is a fraction board devotee, by all means relate $\frac{1}{2}\times\frac{1}{3}$ to the pretty strips of wood.

It would be well if the introduction of, say, formal multiplication of fractions could be meshed for each pupil into the physical situation in which the greatest possible *rapport* exists between pupil and

teacher. After carefully exploiting one type of physical situation with which we associate multiplication of fractions we should turn to a second, and then a third. We *hope* that eventually the student will recognize that multiplication of fractions is not merely cutting up pies or finding areas but rather that it is an operation precisely *defined* by

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

and that this operation has a miraculously wide application to an enormous number of situations.

We have been *describing* the *adjunction* technique as it relates to the extension of the set of natural numbers to include fractions. This is spread over several years of schooling. There is a second method of introducing fractions which is ordinarily seen only in graduate mathematics. This procedure consists of using the numbers 1, 2, $\cdots$ to form symbols (a, b). Seemingly, artificial rules for adding and multiplying these symbols are given, namely

$$(a, b)\cdot(c, d)=(ac, bd)$$
$$(a, b)+(c, d)=(ad+bc, bd)$$

Two of these "numbers" are called *equivalent* and we write $(a, b)\sim(c, d)$ if $ad=bc$.

Now it is proved that these new numbers and the newly defined operations have certain key properties similar to certain ones possessed by the natural numbers. For example

$$[(a, b)+(c, d)]\sim[(c, d)+(a, b)]$$

and

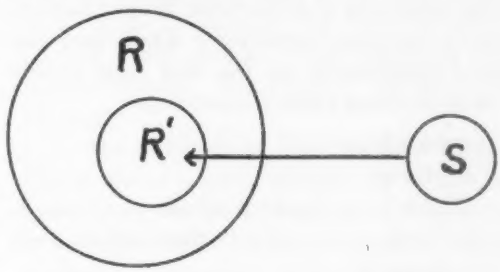
$$(a, b)\cdot[(c, d)+(e, f)]\sim[(a, b)\cdot(c, d)+(a, b)(e, f)].$$

Attention is called to the fact that numbers of the form $(a, 1)$ and $(b, 1)$ act much like numbers a and b, for

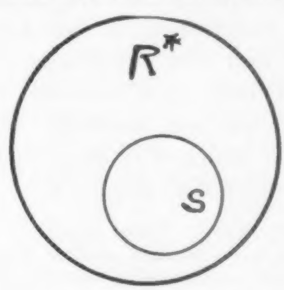
$$(a, 1)+(b, 1)=(a+b, 1);$$
$$(a, 1)\cdot(b, 1)=(ab, 1).$$

And with this observation we have our extension in our hands. We have constructed a system R of number pairs which contains within it a system R' made up of these numbers $(a, 1)$, $(b, 1)$ etc. This system R' has all those and only those properties which characterize the set of numbers $S=[1, 2, 3, \cdots]$. Except for notation it *is* these numbers. We replace the elements of R' by the elements of S and now have S contained in, or as we say, embedded in R. We call R the "set of positive fractions."

We indicate this process symbolically by the diagram below.



This replacement of the elements of R' by the elements of S leaves us with our desired result. Note that in the diagram below we have replaced R by R^*, since R is conceptually distinct from the set which consists of the elements of S and those elements of R which are not in S.



Observe that we end up with the same result whether we proceed by adjunction or by this latter *constructive* method. Which procedure is to be adopted? There is really no choice. Parents *adjoin* some fractions to children before they send them to school. Our textbooks all *adjoin*. One must admit however that the *constructive* technique is *logically* the more

satisfactory. There is no clumsy series of descriptions. No wobbly assumptions are made. For example, when we *adjoin* fractions we may be guilty of arbitrarily *telling* students that $\frac{1}{2}\times\frac{3}{4}=\frac{3}{4}\times\frac{1}{2}$ instead of setting up situations so that the students themselves will discover this fact.

A teacher of arithmetic who sees in a graduate course how the real number system may be erected upon the set of natural numbers, 1, 2, 3, $\cdots$ by constructing firstly, the set of positive fractions, secondly, the set of positive real numbers, and thirdly the set of all real numbers, usually has a deeper insight into the processes of arithmetic. Some teachers even become enthusiastic when they see and understand for the first time proofs of such remarkable theorems as:

$a+b=b+a$

$a+0=a$

There is no fraction whose square is 2.

$a\cdot 0=0$; $-$ if $a\cdot b=0$ then either $a=0$ or $b=0$.

$(-1)\cdot(-1)=1$

$-(-a)=a$

$a\cdot(b+c)=a\cdot b+a.c$

A little contact with the *constructive* method, employed so much in mathematics, will open the teacher's eyes to the many assumptions that are usually made without justification as we *adjoin* new numbers and new operations. Teachers who have been so stimulated will probably use teaching methods which will allow their students to share in the creation of new number concepts.

EDITOR'S NOTE. Those teachers of arithmetic who have studied several good courses in advanced mathematics will appreciate the role of definition and logic in the structure of arithmetic. With all of the areas with which an elementary school teacher must be familiar it is not reasonable to expect more than a small percentage to have mastered the foundations as well as the hierarchies of the mathematics of arithmetic. However, it is necessary that we have a goodly number of competent scholars devoting some time to arithmetic. Most of us teachers ought to recognize the need for good definitions and the role they play in arithmetic. Our brighter pupils in the upper grades frequently have inquiring minds and want to know the "hows" and the "whys" and they are interested in logical structures. We should be able to help them, for it is from this group that we must recruit our future mathematical and scientific scholars.

Multiplying Fractions

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DO YOU have difficulty showing that 2/3 of 3/4 is 1/2? Try this simple device. In the presentation of multiplication of fractions, pupils are usually confused because they are told to multiply numerators and denominators separately for their answers.

To find 2/3 of 3/4, draw a unit square and divide it into fourths horizontally and thirds vertically. This gives 12 blocks. Since we want only 2/3 of something, cross off 1/3 of the blocks and since we actually want 2/3 of 3/4, cross off 1/4 of the blocks remaining. There are then 6 blocks left. These 6 blocks are 6/12 of the original area. Thus 2/3 of 3/4 = 6/12 or 1/2.

With symbols, multiply numerators and denominators separately and we get the 6 and the 12 respectively. These values are the same as were obtained graphically. Reduction and "cancellation" come later.

Factors Determining Attitudes Toward Arithmetic and Mathematics

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In recent months, many scientists, educators and statesmen have referred to the alarming shortage of graduates in engineering, the physical sciences and mathematics. The shortage of persons trained in these fields is being felt in industry, the government and the military services and it is critical in education. A recent article reported:

> "High school principals, faced with an inadequate supply of science and mathematics teachers, have two courses of action open to them. They may use teachers who are inadequately prepared or they may drop courses that their students want and should have. Either action will mean that fewer students in the years ahead will enter college with a developing interest in science and mathematics."[1]

As this article points out, factors in the current shortage seem to be lower salaries of teachers as compared with industry, educational policies designed for mass education, and the fact that scientists have been willing to leave the training of science and mathematics teachers to the teacher training institutions rather than accept or at least share that responsibility.

In addition to the factors which have been suggested as reasons for the current shortage of persons trained in the fields of mathematics and the physical sciences, it is believed that young people may be influenced toward undertaking such studies by their childhood experiences.

The present paper deals with the findings of a preliminary survey of possible factors which might determine students attitudes toward arithmetic and mathematics, based upon the assumption that favorable attitudes toward such subjects affect students' decisions to go into the fields of mathematics, engineering, and the physical sciences.[2] Because an adequate mathematical background is essential for students in such courses, and because it is frequently the mathematical part of these courses which presents the most difficulty, this subject was selected for study.

In the summer of 1955, a pilot study was made of a small sample[3] by means of questionnaires and personal interviews which explored previous influences upon students' attitudes toward arithmetic and mathematics.

The Influence of the Home

The data indicated that parents determine the initial attitudes of their children and affect their achievement in arithmetic and mathematics. Three factors seemed to affect both attitudes and performance in these subjects: parental expectation of children's achievement, parental encouragement regarding these subjects, and parents' own attitudes toward this area of the curricula.

More than half of the students in the sample reported that their parents expected above average work from them in general school achievement but only average work in arithmetic and mathematics. In these cases, students tended

[1] American Association for the Advancement of Science Cooperative Committee on the Teaching of Science and Mathematics. "Improving Science Teaching," *Science*, July 1955, Vol. 122, No. 3160, pp. 145–8.

[2] A study of attitudes as they affect achievement (with ability held constant) is under investigation in a larger study of this subject matter.

[3] Sixteen freshmen enrolled in two summer classes in mathematics at the University of California at Davis.

to do poorly in such subjects. The interviews revealed that the difficulties parents, themselves, had experienced in these subjects were responsible for their lower expectations. In the cases where students remarked that their parents had expected to do good work in these subjects, they tended to like them and do well.[4]

Parental encouragement regarding these subjects and favorable parental attitudes seemed to affect children's interest and achievement positively. Parents who like arithmetic and mathematics themselves are more likely to encourage children and expect good achievement in these subjects. Considering the fact that children tend to identify themselves with their parents and take on their attitudes, it seems that the best way for a child to get an initially positive attitude toward arithmetic and mathematics is to choose parents who like these subjects. The study indicated, however, that even if parents do not like the subjects, a favorable attitude on their part toward the child's studies will affect positively his mental set.

A number of case histories revealed the various ways in which parental remarks and behavior can influence children in their developing interest for arithmetic and mathematics. One young man remarked that his parents had no appreciation for his mathematics courses even though his father was a lawyer and his mother was a college graduate. He said he could recall such negative comments as, "Oh, I'm glad I don't have to go through that again!"

The other examples illustrate how basic attitudes toward arithmetic and mathematics may grow out of the dynamics of family interaction. In both cases, the father's interests were cited as having a determining influence upon the student's desire to become proficient in this area but obviously other factors were involved. One student said he developed a liking for arithmetic at an early age when he found that it was one way he and his accountant father could have something in common, thus successfully competing with his brother for the father's attention. Another student commented, "My father is very good at figures but my mother is not. I can remember my father bawling out my mother for not being able to balance a check book. I guess I just decided I'd be better at it!"

School Environment Summary

Attitudes and performance in arithmetic and mathematics are also affected by teachers, according to the data in the present study. Both favorable and unfavorable experiences were cited, with some student analysis of the problems involved.

One girl said, "I liked arithmetic at first in grammar school, because my mother helped me and made it interesting. I lost interest about the ninth grade when I had a very poor teacher."

A boy indicated that he had had a series of favorable experiences with arithmetic. His teachers in the lower grades had been good and both his father and his mother had encouraged him in mathematics. He said that he had always liked mathematics and science up to his junior year in high school. He reported, "Mathematics would have been my major until I got into second year algebra. I disliked the teacher and the way he taught. I didn't want to work for him. The kids called him 'Little Caesar.' He didn't care whether the kids got the stuff or not. The trouble with him seemed to be a matter of personality rather than lack of knowledge of subject matter." He indicated also that both his parents and adviser had encouraged him to take trigonometry but that he had refused because the same teacher taught this subject.

Another student commented that she had had both good and poor arithmetic

[4] Throughout we have assumed average ability on the part of the child. All of the students in this sample had high school grade averages of "B" or better.

and mathematics experiences: "My problem with arithmetic started in the second grade. I hated the teacher and I wouldn't do anything she suggested. She favored the boys in the class and would play with them. In the fourth grade I had a good teacher. I liked her very much. I had a hard time with multiplication tables but she realized it and helped me.

In referring to her high school experience she said, "In high school, my algebra teacher was good and I liked it very much. She had a good general attitude toward algebra and was willing to help you understand it. You knew that she liked the subject herself. She was a very nice person and always made sure you got the subject." The following year, this student had an unfavorable experience with geometry. "My geometry teacher took the attitude that if you didn't get it, it was too bad. She wasn't going to lose any sleep over it. She was a very smart person because she wrote the book we used. I think it may be hard for someone who knows it so well—they can't understand why someone else can't get it!"

One teacher was cited as being responsible for a student's decision to major in mathematics in college and eventually to teach it in high school. This girl remarked, "In high school I had a very good mathematics teacher. He had a wonderful attitude—and was always willing to talk to you and explore ideas about mathematics. It never seemed like a job to him. He always seemed to enjoy teaching it. He had a terrific personality and was a wonderful person—most human high school teacher I ever had. I often thought that if I became a teacher I'd like to be like him. Seeing how much he liked teaching mathematics, I thought I could also." It is probable that one such teacher can encourage many students to go into the teaching of mathematics.

The above case indicates that the phenomenon of identification is also of importance in the teacher-pupil relationship if positive attitudes are to be acquired toward mathematics. Some interviews indicated, however, that positive attitudes on the part of students are not created as a result of just a liking for the teacher. One student described a popular instructor who was well liked throughout the high school, but who was considered by its students to be a poor teacher. "I lost interest in mathematics my freshman year of high school. I wasn't learning anything in algebra and it wasn't holding my interest. I dreaded the class every day. The teacher was a friend of everyone in the school. All the students knew that they would get good grades even if they didn't do anything in class. The teacher would even help us with the tests! Everyone liked him—he just didn't make the students work. He figured the kids would learn it if they liked him. There was just too much going on in the room."

The same student reported a great contrast in her geometry teacher: "I had a better teacher for geometry. He also taught physics. He put it over so that you could understand it. He had more control of the class. He was very quiet and the class was very quiet. He never had to tell everyone to be quiet—they all knew they should be."

It was interesting to find that no student considered a completely permissive teacher to be a good one. On the other hand, few students cared for overly strict teachers. In a case where a student did prefer rather strict teachers, he indicated that his parents were strict in home discipline.

There was no evidence to indicate that the peer group had much influence upon any individual except perhaps to reinforce attitudes developed from previous experiences. For example, one student reported that he believed his friends had affected his attitude because of their general dislike of mathematics. Further inquiry, however, indicated that this attitude grew from student-body experience with an unpopular teacher.

The present survey indicates that in-

tensive research should be undertaken regarding the development of attitudes toward arithmetic and mathematics. The preliminary study indicated the following points upon which will be based the hypotheses to be tested in continuing research:

(1) Parents determine initial attitudes of their children toward arithmetic and influence their attitudes toward all mathematics.
   (a) Parents who like arithmetic and mathematics tend to convey these positive attitudes to their children.
   (b) Parents who dislike arithmetic and mathematics tend to convey these negative attitudes to their children.

(2) Parents' expectations of their children's performance and the encouragement they give in regard to the study of arithmetic and mathematics affect children's achievement in these areas.
   (a) Assuming ability is present, the children of parents who expect them to do well in arithmetic and mathematics tend to do better than children whose parents expect them to do poorly in such subjects.
   (b) Students whose parents encourage them in their study of arithmetic and mathematics tend to do better and like these subjects more than students whose parents do not encourage them.

(3) Arithmetic and mathematics teachers can have strong positive or negative effects upon students' attitudes and achievement in these areas:
   (a) They build upon attitudes established by parents.
   (b) The enthusiastic teacher leads the students to liking his subject.
   (c) The teachers who tend to affect students' attitudes and achievement positively have the following characteristics: a good knowledge of the subject matter, strong interest in the subject, the desire to have students understand the material, and good control of the class without being overly strict.

EDITOR'S NOTE. Why are some pupils so happy with arithmetic and others so hateful of the subject? Is it the subject, the teacher, the parents, or what combination of these? This preliminary survey shows how important the role of the teacher is in determining the attitudes of pupils. Many good teachers have found ways to develop keen interest in mathematics with pupils whose parents are indifferent and even hostile. Fortunately, most parents are very much interested in the welfare of their children and many want them to do well in arithmetic. There is something wrong in a school where children hate arithmetic because it is a subject that provides social, economic, and psychological appeal. It has the elements of discovery and thinking. It is positive and has a wide range of mental levels. Children can progress and be happy when they are mastering it. It is difficult to assess the value and influence of a good teacher who knows her subject matter and who has the human qualities and who uses methods of teaching which make learning with her both profitable and enjoyable.

The Scientific Method of Problem Solving

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Recently, while engaged in student teaching at The University of Texas, the author worked with a unique method of solving arithmetic word problems which was so successful that she wants to share this technique with other teachers.

While working "thought problems" pupils could see the problem, could see the numbers, but usually were at a loss as to whether they should multiply, divide, add or subtract to arrive at the correct answer. Obviously, they were not "thinking through" the problems and seemed to lack comprehension on the correct procedure to follow for solving the problem. The class had been working on a science unit on electricity in which we followed the scientific method of problem solving in working experiments with electricity. This outline in brief is:

I. Define problem
II. Research
III. Hypothesis
IV. Experiment
V. Conclusion

Using this outline as a basis, an outline was formulated for use in solving arithmetic word problems as follows:

I. What do we want to know?
II. What do we know?
III. Best estimate
IV. Working the problem
V. What did we find out?

After an explanation and discussion of the method the children started solving problems by this process. At first it seemed tedious and time-consuming, but gradually the children became familiar with the process and were able to work more rapidly and with greater understanding and accuracy.

The process seemed to be equally successful with both the slow-learning and the gifted pupils. One child who seemed to be completely bewildered by thought problems, without the remotest idea of where or how to begin, did so well following these steps that he soon became highly proficient in problem solving. Gifted pupils soon were able to solve problems without consciously following the steps in the outline and the method became automatic.

In studying the addition and subtraction of fractions the children made up problems of their own, three of which are presented here:

Problem Number 1: Robert and Larry sold 1/3 of a lot of Christmas trees in one day, 1/8 of the lot the second day and 1/4 of the lot the third day. What part of the lot did they sell in three days?

I. *What do we want to find out?*

What part of the lot did they sell in three days?

II. *What do we know?*

They sold 1/3 of the lot in one day, 1/8 of the lot the second day, 1/4 of the lot the third day.

III. *Best estimate*

They sold about 3/4 of the lot in 3 days.

IV. *Experiment*

$$\begin{aligned} 1/3 &= 8/24 \\ 1/8 &= 3/24 \\ 1/4 &= 6/24 \\ \hline & 17/24 \text{ part of lot sold} \end{aligned}$$

* The author wishes to thank Kay Isbell, Linda Stevens, and Jonny Fruchter, Sixth Grade pupils in Maplewood School, Austin, Texas, for the three problems with their solutions which are presented in this paper.

V. *Conclusion*

They sold 17/24 of the lot in 3 days.

PROBLEM NUMBER 2: Joyce had 1/6 yd. of ribbon but she needed 5/6 yd. to wrap Christmas presents. Ann had 1/2 yd. of ribbon but she needed 3/4 yd. Kay had 4 yds. of ribbon but she only needed 1/2 yd. If Kay, after taking the ribbon she needed, gave Joyce and Ann the ribbon that they needed, would there be any ribbon left over? How much?

I. *What do we want to know?*

Would there be any ribbon left? How much?

II. *What do we know?*

Joyce had 1/6 yd. of ribbon. She needed 5/6 yd. Ann had 1/2 yd. of ribbon. She needed 3/4 yd. Kay had 4 yds. of ribbon. She needed 1/2 yd.

III. *Best estimate*

Yes, there would be about 2 1/2 yds. of ribbon left.

IV. *Experiment*

1/6 yd. = 1/6 yd.
1/2 yd. = 3/6 yd.
4 yd. = 4 yd.
____
4 4/6 = 4 2/3 yd
4 2/3 yd. = 4 8/12 yd.
2 1/12 yd. = 2 1/12 yd.
2 7/12 yd.
5/6 yd. = 10/12 yd.
3/4 yd. = 9/12 yd.
1/2 yd. = 6/12 yd.
____
25/12 = 2 1/12 yd.

V. *Conclusion*

They had 2 7/12 yds. of ribbon left over.

PROBLEM NUMBER 3: An iron smelting company was going to try to smelt 5 tons of iron ore in four days. The first day they smelted 1 1/4 tons of iron ore. The second day they smelted 1 1/3 tons of ore. The third day they smelted 1 1/2 tons of ore. How much more ore did they have to smelt the last day?

I. *What do we want to know?*

How much ore did they have left to smelt the last day.

II. *What do we know?*

They had five tons of ore and were going to try to smelt it in four days. They smelted 1 1/4 tons the first day, 1 1/3 tons the second day, and 1 1/2 tons the third day.

III. *Best estimate*

3/4 of a ton left to smelt the last day.

IV. *Experiment*

1 1/4 tons = 1 3/12
1 1/3 tons = 1 4/12
1 1/2 tons = 1 6/12
____
3 13/12 = 4 1/12 tons
5 tons = 4 12/12 tons
−4 1/2 = 4 1/12 tons
11/12 tons

V. *Conclusion*

They had 11/12 of a ton to smelt the last day.

The idea of helping children see specific steps in problem solving is not new. The value of the scientific method of problem solving lies in relating the work of one area to another—in this instance, science to arithmetic. Experience in applying the scientific method of problem solving to arithmetic proved successful because both the teacher and the children enjoyed doing it.

EDITOR'S NOTE. Although the scientific method or approach to problem solving may be very old, each new generation should discover it. Mrs. Brewer's pupils did this by simple transfer from work in science. There are many opportunities for "tieing together" arithmetic with other areas of learning. Each of these ties tends to give greater understanding and significance to each of the areas involved. The most important element in Mrs. Brewer's development is the role of the pupils in thinking.

A Fraction Circle

Donald B. Lyvers
Parley Coburn School, Elmira, New York

Of all the topics presented in the 7th and 8th grade mathematics courses, those dealing with fractions, decimals, and percentage seem to be the most troublesome and the least understood by the student. Often meaningless rules have been committed to memory by the student, but little comprehension has taken place. Occasionally there are students in a class who may add denominators, or insist that 1/4 times 1/2 does not give the same result as 1/2 times 1/4, or become confused that 1/4 divided by 1/8 is 2. Some students do not know or are not sure that 1/3 is larger than 1/4. These are just a few examples of problems common to many classrooms. I believe that many of these problems have their roots in one or more of the following:

1. The basic idea of number and the nature of our decimal system has never been thoroughly understood.
2. The student has been a victim of the ". . . invert the 2nd term and multiply . . . do as I say" type of training.
3. The student has not had the opportunity to discover for himself what a fraction really is and some of the relationships which exist among fractions.

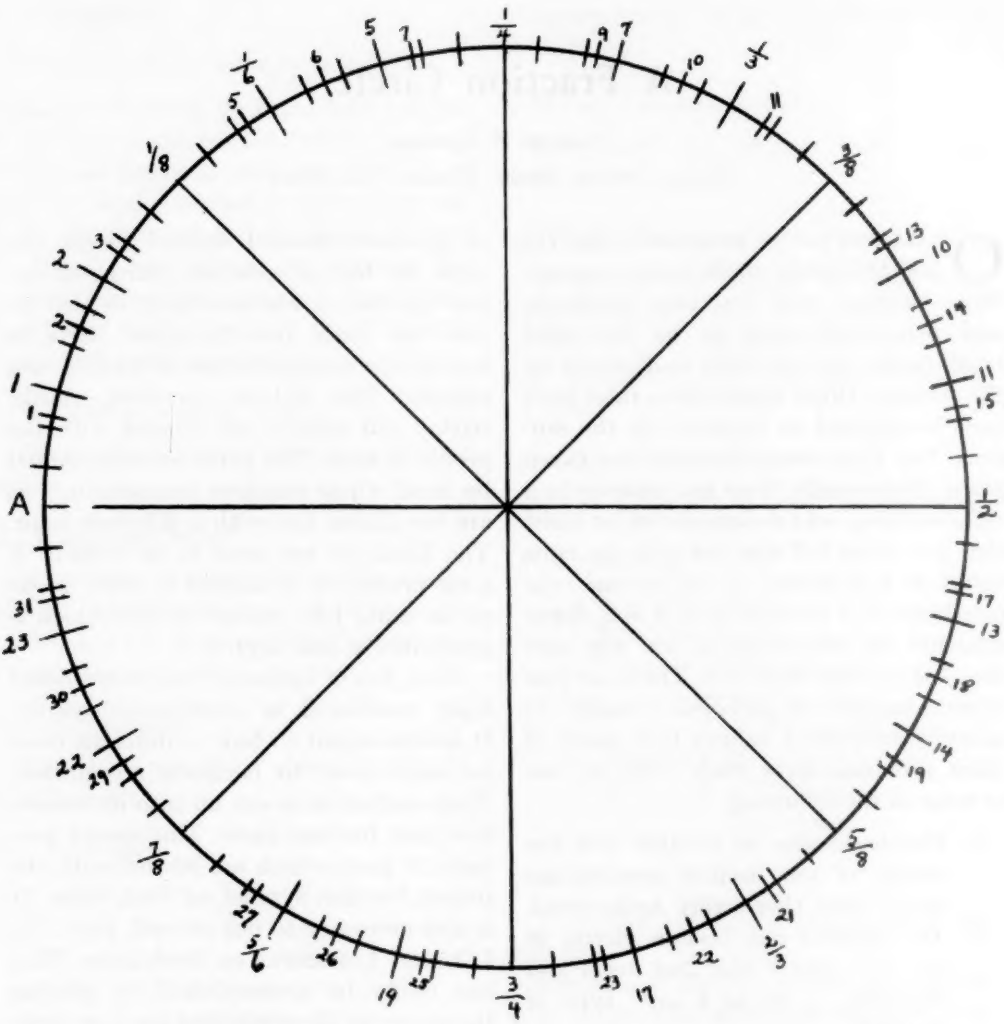
With this third idea in mind, I have made up a device which I hope will enable students to appreciate and understand this business of fractions to a greater degree.

A circle is drawn on a piece of paper or cardboard—a 6″ circle seems to work very nicely. The circle is then divided into 24 equal parts. I found that by dividing each of these parts into thirds would increase the versatility for certain combinations of fractions. Since I wanted to use the circle for halves, quarters, thirds, sixths, and eighths, it was necessary to also divide the circle into 32 equal parts to handle the multiplication of fourths and eighths. The halves, quarters, thirds, sixths, and eighths are labeled with the proper symbol. The 24ths are represented by small whole numbers (numerators), as are the 32nds, but with a different color. The 72nds do not need to be labeled. A good protractor is needed in order to do an accurate job, preferably one which is graduated in half-degrees.

Next, five 6″ circles are cut from colored 6-ply cardboard or construction paper. It is convenient to have a different color for each circle to facilitate recognition. Then each circle is cut up into its respective unit fraction parts. This should produce 23 parts which are labeled with the proper fraction symbol on both sides. It is also necessary to put on each part 1/2, 1/3, and 1/4 marks on both sides. This can easily be accomplished by placing the pieces on the completed fraction circle and marking them accordingly. This completes the construction of the kit.

Arbitrarily, I chose the "9 o'clock" position on the fraction circle as the starting point for the operations of addition, multiplication, and division. The fraction pieces are placed on the circle in a clockwise direction beginning at the radius OA. Suppose we want to add 1/3 and 1/4. The proper fraction pieces are selected and placed on the circle. The two parts added together reach the 14 mark on the circle which is 14/24 or 7/12. More than two fractions with unlike denominators can be added in the same manner, such as 1/3, 1/4, and 1/6.

Subtraction is performed by locating



the minuend on the circle and laying the fraction piece(s) representing the subtrahend in a *counterclockwise* direction from the radius of the minuend. For example, if we wish to subtract 3/8 from 5/6, place three of the 1/8 pieces on the circle in a counterclockwise direction beginning at the 5/6 mark. The result we read off the circle as 11/24.

Division can be accomplished by placing the number of fraction pieces equal to the divisor on the circle in a clockwise direction until the fraction represented by the dividend is reached or *passed*. The number of pieces needed to reach the dividend (it may be a mixed number) is the quotient. Perhaps the problem is to divide 5/6 by 1/3. This requires placing all three 1/3 pieces on the circle. We notice that it takes all of two of the thirds and part of the third piece. We see that 5/6 which is our dividend coincides with the 1/2 mark on the third piece so that our answer is 2 1/2.

Multiplication is a bit different in that the fraction pieces representing either of the two fractions to be multiplied can be placed on the circle, but sometimes one works much better than the other. In the case of 1/6 times 3/4, a piece representing 1/6 can be placed on the circle and then locating the fraction on the circle which

coincides with the 3/4 mark on the piece will give us the answer. Also, placing three 1/4 pieces on the circle and noting that half of the first piece represents 1/6 of the three 1/4 pieces, the fraction coinciding with this mark must also give us the correct solution.

With a little experimentation and imagination, it is possible to handle many combinations in each of the four operations. One of the pleasant features is that most answers can be read off directly, and even when adding and subtracting fractions with unlike denominators, there is no need for finding a common denominator. At most, one may occasionally have to reduce to simplest form.

Provisions for the use of fifths may be added if desired. This could probably best be accomplished by making up another fraction circle divided into fifths, tenths, 20ths and 40ths instead of including them on the circle described above.

It would be ideal for each student in the class to make his own kit, either in part or entirely. All 24 pieces can easily be made from colored 6-ply cardboard which is durable, easily cut, and is readily available at a low cost. The parts can be kept in individual envelopes which would fit in a notebook.

The device described here may be more elaborate than necessary for students who are just beginning the study of fractions. The use of halves, quarters, and eighths may be sufficient. For those who have not yet learned to use compasses and protractor, it may be necessary for the teacher to make up mimeograph or ditto copies of the fraction circle and let the student fill in the fraction symbols. Also the teacher may have to prepare the cardboard or construction paper with the circles and parts already drawn on them, but the student can cut them out. For the more advanced student, the entire kit can probably be made with only a little assistance from the teacher. Just having the student computing the number of degrees required for each of the parts of the circle will prove to be a valuable learning experience for him.

I believe that this project would provide the student with the much needed "discovery" type of learning. The student is able to manipulate fractions physically rather than struggle with meaningless abstract symbols. He can readily "see" that 1/5 is larger than 1/8, that 1/6 is half of 1/3, and that 7/8 divided by 1/6 means how many sixths are contained in 7/8? It must be emphasized that how much a student "discovers" is dependent upon how skillful the teacher is in asking the kind of questions which will foster discovery. Even though this is not intended to be a toy, the "game element" can be used to advantage to stimulate interest.

Two New Courses of Study

These courses arrived too late for competent review but they appear so attractive and inclusive that they are worthy of consultation by teachers and curriculum groups elsewhere. Both are size 8 1/2 by 11 inches and are ring bound and both have approximately 150 pages. The teachers and supervisors who produced them should assure competence of organization and treatment.

Elementary Mathematics, Kindergarten—Grade Six, Bulletin No. 135 for the Public Schools of Montgomery County, Maryland is available from the Board of Education of Montgomery County, Rockville, Maryland. The price is $1.50.

Arithmetic in the Elementary School, a curriculum guide for the Baltimore Public Schools, is published by the Bureau of Publications, Baltimore Public Schools, 3 East 25th St., Baltimore 18, Maryland. The price is $1.00.

BOOK REVIEW

Enrichment Program for Arithmetic, eight pamphlets each for grades 3, 4, 5, and 6, Larsen, Harold D., Row-Peterson, 1956. THE ARITHMETIC TEACHER received the eight booklets for use with grade four. Cost, $1.60 for a set of eight booklets.

These are materials to be used for enrichment. They are suggested as optional materials to be available to all students. The use of these booklets is to be considered a privilege for those people who have completed in a satisfactory manner the regular assigned work. Suggestions are made on the back of each of the eight pamphlets concerning topics with which they might be used to advantage.

Side Trips in Arithmetic, includes discussions of even numbers, odd numbers, ways to discover which whole numbers can be divided without remainders by 2, 3, 4, 5, 6, and 9, short ways to add numbers, and squares of numbers. Explanations made are pictorial as well as verbal. The use of the word "prove" throughout this pamphlet might be criticized. "Show" would serve the purpose.

Find the Number, discusses results of reversing digits in adding, the magic number 9, adding with 7, finding missing numbers, and other sorts of number tricks. The pupils, after some practice, may try these on family, friends, and teachers.

The Story of Zero, tells many interesting facts about this often neglected but most important symbol in our number system. The suggestion of the baseball scoreboard and the bean bag game should interest many little girls and boys. Discussions and pictures describe the use of zero as an indicator of no score on an event, as a place holder, as a remainder, and finally and very interestingly, as a starting point on a scale.

Just Like Magic, includes magic cards, cut out window cards, number cards, mental arithmetic tricks, pattern tricks, and the ever popular age tricks.

Games to Play, includes card type games, Nim, Min, match games, multiplication combination rummy, bingo with multiplication cards, and baseball pitcher games using multiplication combinations.

Magic Squares, has an interesting historic introduction. It includes stars, dominoes, circles and triangles, as well as the more usual construction and completion of magic number squares.

Crossnumber Puzzles, have as subjects addition of tens, subtraction of tens, writing numbers, multiplying with and without carrying, dividing with carrying, and measures.

Time, considers methods of time-telling: sun, shadow, sundial, fire clock, water clock, hour glass, clock with weights, spring driven clock, electric clock, and even a clock using atomic energy. The word "escapement" is not well explained in the discussion of parts of a clock.

This series of pamphlets will be of use to any teacher of a fourth grade. It should give him some new ideas for encouraging pupils to work on their own and also give him some concrete materials to put directly into the hands of the more capable. Mr. Larsen has made a distinct contribution to the early training of the future scientist, engineer, and citizen. A cheer for those who are considering now, material to encourage their bright students. Are you?

DOROTHY MAY SWAN